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Abstract

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MATHEMATICS

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ON THE STABILITY OF THE ROOT NUMBER OF AN ANALYTIC OPERATOR-FUNCTION AND ON PERTURBATIONS OF ITS CHARACTERISTIC NUMBERS AND EIGENVECTORS

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In the present note, the theorem of I. Ts. Gohberg and M. G. Krein ((¹), Theorem 4.3) on the stability of the root number of a linear operator and the theorem on analytic perturbations of characteristic numbers and eigenvectors of a linear operator (see, for example, (²)) are extended to the case of an analytic operator-function.

1. Let G be some domain in the complex plane, and let $A(\lambda)$ be an operator-function analytic in the domain G , whose values are bounded linear operators acting in a Banach space $\mathfrak{B}$. A number $\lambda_0 \in G$ is called a **characteristic number** of the operator-function $A(\lambda)$ if the equation $A(\lambda_0)\varphi = 0$ has a nontrivial solution φ_0 , and the vector φ_0 is called an **eigenvector** of the operator-function $A(\lambda)$ corresponding to λ_0 . The number $\alpha(A(\lambda_0)) = \dim \mathfrak{Z}(A(\lambda_0))$, where $\mathfrak{Z}(A(\lambda_0))$ is the subspace of all eigenvectors corresponding to λ_0 , will be called the **geometric multiplicity** of the characteristic number λ_0 . A set of vectors $\varphi_1, \varphi_2, \dots, \varphi_k$ is called a **chain of vectors associated** with the eigenvector φ_0 of the operator-function $A(\lambda)$, corresponding to the number λ_0 , if

$$\sum_{q=0}^j \frac{1}{q!} A^{(q)}(\lambda_0) \varphi_{j-q} = 0 \quad (j = 1, 2, \dots, k).$$

The number $k + 1$ is called the **length** of this chain. By the multiplicity of the eigenvector φ_0 we shall mean the number $m(\varphi_0)$, equal to the maximal length of a chain of vectors associated with φ_0 . A **canonical system of eigenvectors and associated vectors** of the operator-function $A(\lambda)$, corresponding to the number λ_0 , is a system $\varphi_{j0}, \varphi_{j1}, \dots, \varphi_{j, m_j-1}$ ($j = 1, 2, \dots, \alpha = \alpha(A(\lambda_0))$), where φ_{j0} is an eigenvector of multiplicity m_j , and $\varphi_{j1}, \varphi_{j2}, \dots, \varphi_{j, m_j-1}$ is a chain of

vectors associated with it, with m_1 being the maximal of the multiplicities of all eigenvectors corresponding to λ_0 , and m_j ($j = 2, 3, \dots, \alpha$) being the maximal of the multiplicities of all eigenvectors from some direct complement in $\mathfrak{Z}(A(\lambda_0))$ to the subspace spanned by $\varphi_{10}, \dots, \varphi_{j-1,0}$. The number

$$m(\lambda_0, A) = \sum_{j=1}^{\alpha} m_j$$

is called the **algebraic multiplicity** of the characteristic number λ_0 of the operator-function $A(\lambda)$. A point λ will be called a **regular point** of the operator-function $A(\lambda)$ if the operator $A(\lambda)$ is continuously invertible, and a **Φ -point** of the operator-function $A(\lambda)$ if the operator $A(\lambda)$ is a Φ -operator (see ⁽¹⁾, p. 52).

Let Γ be a closed rectifiable contour situated in the domain G and having, with respect to the operator-function $A(\lambda)$, the following properties:

- a) inside the contour Γ there is a finite number of characteristic numbers $\lambda_1, \lambda_2, \dots, \lambda_s$ of the operator function $A(\lambda)$, which are its Φ -points;
- b) all the remaining points λ ($\lambda \neq \lambda_j, j = 1, 2, \dots, s$) situated inside the contour and on the contour Γ are regular points of the operator function $A(\lambda)$.

From Theorem 1 of A. S. Markus ⁽³⁾ it follows that $m(\lambda_j, A) < \infty$ ($j = 1, 2, \dots, s$). The **root number** of the operator function $A(\lambda)$, **corresponding to the contour** Γ , will mean the sum of the algebraic multiplicities of all characteristic numbers λ_j ($j = 1, 2, \dots, s$) of the operator function $A(\lambda)$ located inside the contour Γ , i.e. the number

$$m(A, \Gamma) = m(\lambda_1, \Gamma) + m(\lambda_2, \Gamma) + \dots + m(\lambda_s, A).$$

Theorem 1. *Let Γ be a closed rectifiable contour bounding a domain G , and let $A(\lambda)$ be an operator function, analytic in G and continuous in the closed domain $G \cup \Gamma$, with the contour Γ possessing, relative to the operator function $A(\lambda)$, properties a), b). Then there exists a number $\delta > 0$ such that, whatever the operator function $B(\lambda)$, analytic in G , continuous in $G \cup \Gamma$, and satisfying the condition*

$$\max_{\lambda \in \Gamma} \|B(\lambda) - A(\lambda)\| < \delta, \quad (1)$$

the contour Γ possesses properties a), b) also relative to $B(\lambda)$, and moreover

$$m(B, \Gamma) = m(A, \Gamma).$$

For the proof of Theorem 1 the following two lemmas are needed:

Lemma 1. Let $\{A_j\}_0^\infty$ be a sequence of linear bounded operators acting in a Banach space $\mathfrak{B}$ and satisfying the condition $\sup \|A_j\| < \infty$, and

$$A(\lambda) = \sum_{j=0}^{\infty} \lambda^j A_j \quad (|\lambda| < 1).$$

Let, further, $\widetilde{\mathfrak{B}}$ be the Banach space of all sequences $\tilde{x} = \{x_j\}_1^\infty$ of elements from $\mathfrak{B}$ satisfying the condition

$$\|\tilde{x}\| = \sum_{j=1}^{\infty} \|x_j\| < \infty,$$

and let $\widetilde{A}_0, \widetilde{C}$ be the operators defined in $\widetilde{\mathfrak{B}}$ by the matrices

$$\widetilde{A}_0 = \begin{pmatrix} A_0 & A_1 & \dots & A_n & \dots \\ 0 & I & \dots & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & I & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}, \quad \widetilde{C} = \begin{pmatrix} 0 & 0 & \dots & 0 & \dots \\ -I & 0 & \dots & 0 & \dots \\ 0 & -I & \dots & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}.$$

A set of vectors $\widetilde{\varphi}_j = \{\varphi_j^{(k)}\}_{k=0}^\infty$ ($j = 0, 1, \dots, p$) is then and only then a chain of eigenvectors and associated vectors of the linear pencil $\widetilde{A}(\lambda) = \widetilde{A}_0 + \lambda \widetilde{C}$, corresponding to the characteristic number λ ($|\lambda| < 1$), when $\varphi_j^{(0)}$ ($j = 0, 1, \dots, p$) is a chain of eigenvectors and associated vectors of the operator function $A(\lambda)$, corresponding to the characteristic number λ , and

$$\varphi_0^{(k)} = \lambda \varphi_0^{(k-1)} \quad (k = 1, 2, \dots, \infty); \quad \varphi_j^{(k)} = \lambda \varphi_j^{(k-1)} + \varphi_{j-1}^{(k-1)}$$

$$(j = 1, 2, \dots, p; k = 1, 2, \dots, \infty).$$

Lemma 2. Let $A(\lambda)$ and $B(\lambda)$ be operator-functions, analytic inside the circle $|\lambda| < r$ and continuous in the closed circle $|\lambda| \leq r$, and let Γ be a closed rectifiable contour lying inside the circle $|\lambda| < r$. If the contour Γ has, with respect to the operator-function $A(\lambda)$, properties a), b), then there exists a number $\delta > 0$ such that, when the condition

$$\max_{|\lambda|=r} \|B(\lambda) - A(\lambda)\| < \delta$$

is satisfied, the contour Γ has properties a), b) also with respect to the operator-function $B(\lambda)$, and moreover

$$m(B, \Gamma) = m(A, \Gamma).$$

Lemma 1 is a generalization of a well-known property of polynomial operator pencils (see (4), p. 326). Lemma 2 is proved on the basis of Lemma 1 and the theorem from (5).

Proof of Theorem 1. Let $\lambda_1, \lambda_2, \dots, \lambda_s$ be all the characteristic numbers of the operator-function $A(\lambda)$ in the domain G (without regard to multiplicity). Choose numbers $\varepsilon_j > 0$ ($j = 1, 2, \dots, s$) such that the disks $G_j = \{\lambda : |\lambda - \lambda_j| < \varepsilon_j\}$ lie in G and are pairwise disjoint. Put $\gamma_j = \{\lambda : |\lambda - \lambda_j| = \varepsilon_j/2\}$. By Lemma 2, for each $j = 1, 2, \dots, s$ there exists a number $\delta_j > 0$ such that from the inequality

$$\max_{|\lambda - \lambda_j| = \varepsilon_j} \|B(\lambda) - A(\lambda)\| < \delta_j \quad (2)$$

it follows that the contours γ_j have, with respect to $B(\lambda)$, properties a), b), and moreover

$$m(B, \gamma_j) = m(A, \gamma_j). \quad (3)$$

Denote by F the closed set obtained by removing from $G \cup \Gamma$ the open disks $|\lambda - \lambda_j| < \varepsilon_j/2$ ($j = 1, 2, \dots, s$). For $\lambda \in F$ the operator $A(\lambda)$ is invertible. Put

$$M = \max_{\lambda \in F} \|A^{-1}(\lambda)\|.$$

We now show that the number

$$\delta = \min(\delta_1, \delta_2, \dots, \delta_s, 1/M)$$

has the required property. Indeed, if inequality (1) is satisfied, then for any point $\lambda \in F$

$$\|B(\lambda) - A(\lambda)\| < \delta \leq 1/M \leq 1/\|A^{-1}(\lambda)\|,$$

and therefore the operator $B(\lambda)$ is invertible.

On the other hand, from (1) it follows that, for all $j = 1, 2, \dots, s$, (2) holds, and therefore all contours γ_j have, with respect to $B(\lambda)$, properties a), b), and the equalities (3) hold.

From what has been said above it follows directly that the contour Γ has properties a), b) with respect to the operator-function $B(\lambda)$ and

$$m(B, \Gamma) = \sum_{j=1}^s m(B, \gamma_j) = \sum_{j=1}^s m(A, \gamma_j) = m(A, \Gamma).$$

The theorem is proved.

2. Theorem 2. Let $\{A_j(\varepsilon)\}_0^\infty$ be a sequence of operator-functions, analytic in some disk $|\varepsilon| < \delta$ and satisfying the condition

$$\sup_j \|A_j(\varepsilon)\| < \infty \quad (|\varepsilon| < \delta);$$

$$A_\varepsilon(\lambda) = \sum_{j=0}^{\infty} A_j(\varepsilon) \lambda^j \quad (|\lambda| < 1, |\varepsilon| < \delta).$$

Let Γ be a closed rectifiable contour lying in the circle $|\lambda| < 1$, inside which there is a single characteristic number λ_0 of the operator-function $A_0(\lambda)$, of finite algebraic multiplicity m . Then:

- 1) There exists a number ρ ($0 < \rho \leq \delta$) and a natural number r ($\leq m$) such that, for $0 < |\varepsilon| < \rho$, the operator-function $A_\varepsilon(\lambda)$ has exactly r distinct characteristic numbers inside the contour Γ . These characteristic numbers can be arranged into groups $\lambda_{ij}(\varepsilon)$ ($i = 1, 2, \dots, l$; $j = 1, 2, \dots, p_i$; $\sum_{i=1}^l p_i = r$) in such a way that the functions of one group (i.e.

$$\lambda_{i1}, \lambda_{i2}, \dots, \lambda_{ip_i})$$

constitute the complete set of branches of a p_i -valued function; moreover, in the circle $|\varepsilon| < \rho$ they are represented in the form of series

$$\lambda_{ij}(\varepsilon) = \lambda_0 + \sum_{k=1}^{\infty} a_{ik} \left[(\varepsilon^{1/p_i})_j \right]^k \quad (j = 1, 2, \dots, p_i; i = 1, 2, \dots, l),$$

where $(\varepsilon^{1/p_i})_j$ ($j = 1, 2, \dots, p_i$) is the complete set of branches of the function ε^{1/p_i} .

- 2) There are numbers m_i ($i = 1, 2, \dots, l$) such that $m(\lambda_{ij}(\varepsilon), A_\varepsilon) = m_i$ ($j = 1, 2, \dots, p_i$; $0 < |\varepsilon| < \rho$), and

$$\sum_{i=1}^l \sum_{j=1}^{p_i} m(\lambda_{ij}(\varepsilon), A_\varepsilon) = \sum_{i=1}^l p_i m_i = m.$$

3) There are numbers $\alpha_i \leq \alpha(A_0(\lambda_0))$ ($i = 1, 2, \dots, l$) such that

$$\alpha(A_\varepsilon(\lambda_{ij})) = \alpha_i \quad (j = 1, 2, \dots, p_i; 0 < |\varepsilon| < \rho).$$

4) A basis of the eigenspace $\mathfrak{Z}(A_\varepsilon(\lambda_{ij}(\varepsilon)))$ can be represented, for $0 < |\varepsilon| < \rho$, in the form of series

$$\varphi_{ij}^{(q)}(\varepsilon) = \sum_{k=0}^{\infty} \varphi_{ik}^{(q)} [(\varepsilon^{1/p_i})_j]^k \quad (q = 1, 2, \dots, \alpha_i),$$

where $\varphi_{ik}^{(q)}$ are certain vectors, and $\varphi_{i0}^{(q)} \in \mathfrak{Z}(A_0(\lambda_0))$.

The proof of this theorem is carried out analogously to the proof of Theorem 2 of [6], with the aid of Lemma 1.

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REFERENCES

1. I. Ts. Gokhberg, M. G. Krein, *UMN*, 12, no. 2 (1957).
2. T. Kato, *J. Math. Soc. Japan*, 4 (1952).
3. A. S. Markus, *DAN*, 119, no. 6 (1958).
4. I. Ts. Gokhberg, M. G. Krein, *Introduction to the Theory of Linear Nonselfadjoint Operators in Hilbert Space*, "Nauka," 1965.
5. V. M. En, *Izv. AN MSSR*, no. 4 (1966).
6. V. M. En, *Mathematical Investigations*, 1, issue 1, Kishinev, 1966.

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