

ON THE QUESTION OF CONFORMAL MAPPING OF CIRCULAR DOMAINS

B. G. Baibarin

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Abstract

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MATHEMATICS

B. G. BAIBARIN

ON THE QUESTION OF CONFORMAL MAPPING OF CIRCULAR DOMAINS

(Presented by Academician M. A. Lavrent'ev on 28 III 1966)

1. Let L be a simple nonclosed curve composed of a finite number of arcs of circles and line segments, beginning at a point $\zeta_0 > 0$ of the real axis, not passing through the origin, and not intersecting the ray $l: \zeta_0 < z < \infty$.

Notation.

1)

$$z = \Phi(w, t), \quad \Phi(\beta(t), t) = 0, \quad \Phi(\infty, t) = \infty, \quad (1)$$

is a function conformally mapping the half-plane $\text{Im } w > 0$ onto the domain $B(t)$, obtained from the plane (z) by making a slit along the curve $\mathcal{L}(t) + l$, where $\mathcal{L}(t): z = \xi(\tau)$, $t_0 \leq \tau \leq t$, $t < T$, is a part of the curve L .

- 2) $a_p(t)$, $p = 0, 1, 2, \dots, n$, are the preimages of the corner points z_i , $i = 0, 1, \dots, m$; $n = 2m$; $z_0 = \zeta_0$ of the curve $\mathcal{L}(t) + l$, including also the end of the slit, whose preimage we consider to be $a_0(t)$ (to each non-end corner point z_i there correspond two points among the $a_p(t)$) under the mapping (1).
- 3) $\alpha_p\pi$, $p = 0, 1, 2, \dots, n$, is the angle between the tangents on the left and on the right to the curve $\mathcal{L}(t) + l$ at the points z_i .
- 4) $M_p(t)$, $p = 0, 1, 2, \dots, n$, are the accessory constants of the Schwarz derivative $S(w, t)$ for the function $\Phi(w, t)$ (3).
- 5) t_i , $i = 0, 1, 2, \dots, m$, are the values of the parameter t corresponding to the corner points z_i , $z_i = \xi(t_i)$.

Theorem. Let $\chi(w)$ be a function conformally mapping the half-plane $\text{Im } w > 0$ onto the plane with a slit along the ray l . Then the functions

$$\Psi(w, t) = \log \Phi'_w(w, t), \quad S(w, t) = \Psi''(w, t) - \frac{1}{2} \Psi'^2(w, t)$$

are integrals of the equations

$$\frac{\partial \Psi(w, t)}{\partial t} + \frac{1}{w - a_0(t)} \frac{\partial \Psi(w, t)}{\partial w} = \frac{1}{(w - a_0(t))^2}, \quad (2)$$

$$\frac{\partial S(w, t)}{\partial t} + \frac{1}{w - a_0(t)} \frac{\partial S(w, t)}{\partial w} - \frac{2S(w, t)}{(w - a_0(t))^2} - \frac{6}{(w - a_0(t))^4} = 0, \quad (3)$$

satisfying the initial conditions

$$\Psi(w, t_0) = \log \chi'(w), \quad S(w, t_0) = \Psi''(w, t_0) - \frac{1}{2} \Psi'^2(w, t_0).$$

2. Denote by $\omega(t, t_m)$ the integral of the equation

$$\frac{d\omega(t, t_m)}{dt} = \frac{1}{\omega(t, t_m) - a_0(t)}, \quad (4)$$

which for $t = t_m$ takes the value $a_0(t_m)$. Then equation (4) has:

a) a unique integral $\omega_1(t, t_m)$, $\omega_1(t_m, t_m) = a_0(t_m)$, for which

$$\lim_{t \rightarrow t_m + 0} \frac{\omega_1(t, t_m) - a_0(t)}{\sqrt{t - t_m}} = \sqrt{\frac{2\alpha_1}{\alpha_n}};$$

b) a unique integral $\omega_2(t, t_m)$, $\omega_2(t_m, t_m) = a_0(t_m)$, for which

$$\lim_{t \rightarrow t_m + 0} \frac{\omega_2(t, t_m) - a_0(t)}{\sqrt{t - t_m}} = -\sqrt{\frac{2\alpha_n}{\alpha_1}}.$$

Moreover, the relation

$$\lim_{t \rightarrow t_m + 0} \frac{\omega_1(t, t_m) - a_0(t)}{\omega_2(t, t_m) - a_0(t)} = -\frac{\alpha_1}{\alpha_n}$$

is valid.

3. The curvature $\varkappa(t)$ of the curve $\mathcal{L}(t) + l$ at the point $z = \zeta(t)$ is determined by the formula

$$s'(t)\varkappa(t) = \operatorname{Im} \left\{ i \frac{a_0(t) - \alpha(t)}{\gamma(t)} + \left(\frac{d\bar{\beta}(t)}{dt} \right)^2 - \frac{d\bar{\beta}(t)}{dt} \frac{da_0(t)}{dt} + \frac{1}{\pi} \int_{t_0}^t \frac{\partial H^*(a_0(t), t, \tau)}{\partial t} d\theta(\tau) \right\}, \quad (5)$$

where $s(t)$ is the length of the curve $\mathcal{L}(t)$, measured from the point ζ_0 ; $\theta(t)$ is the angle formed by the tangent to the curve $\mathcal{L}(t)$ at the point $z = \zeta(t)$ with

the real axis; $a(t) = \operatorname{Re} \beta(t)$, $\gamma(t) = \operatorname{Im} \beta(t)$; $\beta(t)$ is the integral of equation (4), assuming at $t = t_0$ the value $\beta(t_0) = \beta_0$, $\operatorname{Im} \beta_0 > 0$, and

$$H^*(a_0(t), t, \tau) = \log \frac{a_0(t) - \omega_2(t, \tau) \omega_1(t, \tau) - \beta(t)}{a_0(t) - \omega_1(t, \tau) \omega_2(t, \tau) - \beta(t)}.$$

4. What has been said above, as well as what was set forth in (1), makes it possible to propose a method for determining the parameters of the Schwarz derivative $S(w)$ for the function $z = \Phi(w)$, which conformally maps the half-plane $\operatorname{Im} w > 0$ onto a simply connected bounded domain B , whose boundary Γ consists of a finite number of arcs of circles and line segments. It may be assumed, without loss of generality, that the domain B contains the origin and that one of the corner points of the boundary Γ (denote it by $z_0 = \zeta_0$) lies on the positive part of the real axis. We now consider a one-parameter family of domains $B(t)$, obtained from the (z) -plane by making the cut $\mathcal{L}(t) + l$, first along the real axis from $+\infty$ to the point z_0 , and then along the curve Γ to the point $z = \zeta(t)$. As $t \rightarrow T$, the family $B(t)$ converges to the domain B as to a kernel. Let the corner points z_i , beginning with z_0 , be numbered so that a point with a larger index follows a point with a smaller index under positive traversal of the domain B along the boundary Γ . Suppose, further, that the function $z = \Phi(w, t_k)$, conformally mapping the half-plane $\operatorname{Im} w > 0$ onto the domain $B(t_k)$, is known. Then the preimages $a_{p0}^{(k)}$, $p = 0, 1, 2, \dots, 2k-1$, of the corner points of the boundary of the domain $B(t_k)$, and the accessory constants $M_{p0}^{(k)}$, $p = 0, 1, 2, \dots, 2k-1$, of the Schwarz derivative $S(w, t_k)$ are also known. We now extend the cut $\mathcal{L}(t_{k+1}) + l$ to the next corner point of the curve Γ . Then the preimages $a_p^{(k+1)}(t_{k+1})$, $p = 0, 1, 2, \dots, 2k+1$, of the corner points of the curve $\mathcal{L}(t_{k+1}) + l$ under the mapping $z = \Phi(w, t_{k+1})$, and also the accessory constants $M_p^{(k+1)}(t_{k+1})$, $p = 0, 1, 2, \dots, 2k+1$, of the Schwarz derivative $S(w, t_{k+1})$ are determined as integrals of the system (3) from (1), p. 13, satisfying the initial conditions $a_p^{(k+1)}(t_k) = a_{p0}^{(k)}$, $p = 2, 3, \dots, 2k$; $a_j^{(k+1)}(t_k) = a_{00}^{(k)}$, $j = 0, 1, 2k+1$; $M_p^{(k+1)}(t_k) = M_{p0}^{(k)}$, $p = 2, 3, \dots, 2k$;

$$\lim_{t \rightarrow t_k+0} (t - t_k) M_j^{k+1}(t), \quad j = 0, 1, 2k+1.$$

The system (3) from (1) is integrated by some numerical method (2) from the value of the parameter $t = t_k$ to the value $t = t_{k+1}$. The latter, in turn, is determined from the relation

$$s(t_{k+1}) = \int_{t_0}^{t_{k+1}} \left\{ \frac{1}{4} |\gamma(t)|^{-3} |a_0^{(k+1)}(t) - \beta(t)|^3 \exp \left[-2 \int \left(\frac{d\gamma}{dt} \right)^2 dt \right] \times \right.$$

$$\times \exp \frac{1}{\pi} \int_{t_0}^t \operatorname{Re} H^*(a_0^{(k+1)}(t), t, \tau) d\theta(\tau) \Big\} 2 \left(\frac{d\gamma}{dt} \right)^2 dt.$$

For the start of the computation one may use the series

$$a_p^{(k+1)}(t) = a_{p0}^{(k)} + a_{p1} \sqrt{t-t_k} + a_{p2} \left(\sqrt{t-t_k} \right)^2 + \dots, \quad p = 0, 1, 2, \dots, 2k+1;$$

$$M_p^{(k+1)}(t) = M_{p0}^{(k)} + m_{p1} \sqrt{t-t_k} + m_{p2} \left(\sqrt{t-t_k} \right)^2 + \dots,$$

$$p = 2, 3, \dots, 2k;$$

$$M_j^{(k+1)}(t) = \frac{m_{j,-1}}{\sqrt{t-t_k}} + m_{j0} + m_{j1} \sqrt{t-t_k} + \dots; \quad j = 0, 1, 2k+1,$$

in which the functions $a_p^{(k+1)}(t)$, $M_p^{(k+1)}(t)$ are expanded. Thus, the constants $a_p^{(k+1)}(t_{k+1})$ and $M_p^{(k+1)}(t_{k+1})$ will be determined with the accuracy allowed by the numerical method. Noting further that the function $z = \Phi(w, t_0)$, mapping the half-plane $\operatorname{Im} w > 0$ onto the plane with a slit along the ray $l: z_0 \leq z < \infty$, is known, and that the family of functions $\Phi(w, t)$, as $t \rightarrow T$, converges uniformly inside the domain $\operatorname{Im} w > 0$ to $\Phi(w)$, we conclude that, by the method described above, all the parameters entering Schwarz' s equation (3) for the function $z = \Phi(w)$ can be determined.

Novosibirsk Electrotechnical
Institute

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.

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