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Abstract

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MATHEMATICS

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ON THE THEORY OF DISTAL MINIMAL SETS AND DISTAL FUNCTIONS

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1°. Let (X, ρ) be a compact metric space; J the group of real numbers; (X, J, π) a dynamical system ⁽¹⁾ (a group of transformations ⁽²⁾). The system (X, J, π) is called **distal** ⁽³⁾ if, for any two distinct points x and y of X , there is a number $d > 0$ such that $\rho(xt, yt) > d$ for all $t \in J$. Every equicontinuous ⁽²⁾ (uniformly Lyapunov stable ⁽¹⁾) system is, obviously, distal.

In the work of Auslander, Hahn, and Markus ⁽⁴⁾ it was shown that on every nilmanifold ⁽⁵⁾ X one can define a dynamical system (X, J, π) under which X is a distal minimal set, but, generally speaking, is not equicontinuous.

In the present note it is proved that on every torus T^α (finite-dimensional or infinite-dimensional) one can define a distal dynamical system under which T^α is a minimal set. An example is given of a distal non-equicontinuous system on the three-dimensional torus T^3 , under which T^3 is a minimal set. Further, it is proved that if $\varphi(t)$ and $\psi(t)$ ($t \in J$) are Bohr almost-periodic functions ⁽⁶⁾, then the function

$$\varphi \left(\int_0^t \psi(\tau) d\tau \right)$$

is distal ^(7,8), but, generally speaking, is not Bohr almost-periodic.

2°. In what follows, by T we shall denote the factor group of the group J by the subgroup generated by the number 2π .

Theorem 1. *A dynamical system defined on the n -dimensional torus T^n by the system of differential equations*

$$dx_1/dt = \varphi_1,$$

$$dx_i/dt = \varphi_i(x_1, \dots, x_{i-1}) \quad (i = 2, \dots, n), \quad (1)$$

where $\varphi_i \in J$, $\varphi_i(x_1, \dots, x_{i-1})$ are functions periodic with period 2π in each variable and is distal. For the system (1) there exist numbers $\gamma_2, \dots, \gamma_n$ such that the system of equations

$$dx_1/dt = \varphi_1,$$

$$dx_i/dt = \varphi_i(x_1, \dots, x_{i-1}) + \gamma_i \quad (i = 2, \dots, n) \quad (1')$$

defines a dynamical system under which T^n is a minimal set.

Proof. The proof of Theorem 1 is carried out by induction on n . For $n = 1$ the assertion is obvious. Suppose the assertion is true for $n = k$, and let $a = (x_1, \dots, x_{k+1})$ and $b = (y_1, \dots, y_{k+1})$ be two distinct points of the torus T^{k+1} . If there is an index i such that $1 \leq i \leq k$ and $x_i \not\equiv y_i \pmod{2\pi}$, then the points a and b are distal by virtue of the induction hypothesis. If, however, $x_i \equiv y_i \pmod{2\pi}$ for all i , $1 \leq i \leq k$, then $x_{k+1} \not\equiv y_{k+1} \pmod{2\pi}$, and in this case the distance between the points at and bt does not depend on $t \in J$.

By virtue of the induction assumption we may suppose that the system of equations (1) defines on the k -dimensional torus $T^k = \{(x_1, \dots, x_k)\}$ a dynamical system for which T^k is a distal minimal set. It is easy to prove that the intersection of the closure of the trajectory of the point $p = (0, \dots, 0) \in T^{k+1}$ with the circle $(0, \dots, 0, x_{k+1})$ is a subgroup of this group. To complete the proof of the second assertion of the theorem it suffices to use the following proposition.

Lemma. Let X be a compactum and let p be a homomorphic mapping of a distal dynamical system (X, J, π) onto a system (Y, J, ρ) , in which Y is a minimal set. If there exists a point $x \in X$ such that the closure of the trajectory of the point x contains the set $p^{-1}(p(x))$, then X is a minimal set for (X, J, π) .

Theorem 1 can obviously be generalized also to infinite-dimensional tori.

We now show that on any torus T^n ($n \geq 3$) one can define a dynamical system for which T^n is a distal, not uniformly continuous, minimal set. By virtue of Theorem 1 it suffices to construct an example of such a system on T^3 .

Example. In ⁽¹⁾, on p. 424, it is shown that there exist an irrational number μ and a function $\Phi(x_1, x_2)$, continuous on T^2 , such that for any $m > 0$ and $\varepsilon > 0$ one can specify $a \in T$ and $t \in J$ for which

$$\left| \int_0^t \Phi(\tau, \mu\tau) d\tau - \int_0^t \Phi(\tau, a + \mu\tau) d\tau \right| > m, \quad |a| < \varepsilon \pmod{2\pi}. \quad (2)$$

By virtue of Theorem 1 there exists a number γ such that the system of equations

$$\begin{aligned} dx_1/dt &= 1, \\ dx_2/dt &= \mu, \\ dx_3/dt &= \Phi(x_1, x_2) + \gamma \end{aligned} \quad (3)$$

defines on T^3 a distal dynamical system for which T^3 is a minimal set. From condition (2) it follows that the system (3) is not uniformly continuous.

From the results of the paper ⁽⁹⁾ the following assertion follows.

Theorem 2. If the torus T^2 is a distal minimal set for the system (T^2, J, π) , then this system is uniformly continuous.

However, the torus T^2 may be a distal, not uniformly continuous, minimal set for a discrete system.

3°. A continuous real function $\varphi(t)$ ($t \in J$) is called **distal** ^(7, 8) if the closure of the trajectory of the point $\varphi(t)$ in the Bebutov system ⁽¹⁰⁾ is a compact distal minimal set. It is known ⁽⁷⁾ that the set of all distal functions forms an algebra. It is easy to prove that the limit of a uniformly convergent sequence of distal functions is a distal function. From ⁽¹¹⁾ it follows that, for any distal function $\Phi(t)$, the limit

$$\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^{+T} \Phi(t+a) dt$$

exists uniformly in $a \in J$.

Denote by B the set of all Bohr almost-periodic functions.

Theorem 3. If $\psi \in B$, then the set

$$B_\psi = \left\{ \varphi \left(\int_0^t \psi(\tau) d\tau \right) \mid \varphi \in B \right\}$$

forms an algebra of distal functions.

Proof. It is known ⁽⁶⁾ that for every function $\psi \in B$ there exist a function $F(x_1, \dots, x_n, \dots)$, continuous on T^∞ , and numbers $\lambda_1, \dots, \lambda_n, \dots$ such that

$$\psi(t) = F(\lambda_1 t, \dots, \lambda_n t, \dots).$$

The system

$$\begin{aligned} dx_i/dt &= \lambda_i \quad (i = 1, \dots, n, \dots), \\ dy/dt &= F(x_1, \dots, x_n, \dots) \end{aligned}$$

is distal. Therefore, for any function $\varphi(y)$ periodic with period 2π , the function

$$\varphi \left(\int_0^t F(\lambda_1 \tau, \dots, \lambda_n \tau, \dots) d\tau \right) = \varphi \left(\int_0^t \psi(\tau) d\tau \right)$$

is distal. To complete the proof it suffices to use the remarks made at the beginning of this section.

The example constructed above shows that functions of the form

$$\varphi \left(\int_0^t \psi(\tau) d\tau \right),$$

$(\varphi, \psi \in B)$, generally speaking, are not almost periodic in the sense of Bohr. Moreover, they are not, generally speaking, almost periodic in the sense of Levitan ⁽⁶⁾, as the following assertion shows.

Theorem 4. *If a function almost periodic in the sense of Levitan is distal, then it is almost periodic in the sense of Bohr.*

In conclusion we note that, using Theorem 1, one can also indicate other classes of distal functions.

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