

COMPARISON THEOREMS FOR FRACTIONAL POWERS OF OPERATORS

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Abstract

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MATHEMATICS

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COMPARISON THEOREMS FOR FRACTIONAL POWERS OF OPERATORS

(Presented by Academician I. N. Vekua, June 21, 1966)

1. In this paper a perturbation theory is developed for fractional powers of weakly positive operators acting in a Banach space E (w.p. E -operators)
 *. The results obtained make it possible to establish how the fractional powers of elliptic operators change when the coefficients of the equations and the boundary conditions are changed.
2. Let A_1 and A_2 be w.p. E -operators. Let

$$\Phi(v, w) \equiv (A_1 v_1, w) - (v, A_2^* w) = \sum_{i=1}^m \Phi_i(v, w) \quad (v \in D(A_1), w \in D(A_2^*)). \quad (1)$$

Here (φ, ψ) is the value of the functional ψ from the conjugate E -space E^* ; A_2^* is the operator acting in E^* conjugate to A_2 ; $\Phi_i(v, w)$ are bilinear forms defined on $D(A_1) \times D(A_2^*)$, satisfying, for certain δ_i and ρ_i in $[0, 1]$, the inequalities

$$|\Phi_i(v, w)| \leq R_i \|A_1 v\|^{\delta_i} \|v\|^{1-\delta_i} \|A_2^* w\|^{\rho_i} \|w\|^{1-\rho_i}. \quad (2)$$

Let $-\min \delta_i < s < 1 - \max \delta_i$, if all $\delta_i \in (0, 1)$. If, however, $\min \delta_i = 0$ or $\max \delta_i = 1$, then in the corresponding place the sign $<$ must be replaced by the sign $\leq$. Let τ satisfy the same conditions with respect to the system of numbers ρ_i .

Theorem 1. Let $-1 < \alpha < 1 - \tau - \max \rho_i$. Let the numbers $\Delta_i = \rho_i + \delta_i + \tau + s + \alpha$ be renumbered so that $\Delta_i > 1$ for $i = 1, \dots, l$, $\Delta_i = 1$ for $i = l + 1, \dots, k$, and $\Delta_i < 1$ for $i = k + 1, \dots, m$. Then for any $0 < \varepsilon_i < 1 - \delta_i - s$, $i = l + 1, \dots, k$, and $v \in D(A^\gamma)$, $\gamma = \max(1, \alpha + s)$, the inequality

$$\|A_2^\tau (A_1^\alpha - A_2^\alpha) A_1^s v\| \leq c \left[\sum_{i=1}^l R_i (\Delta_i - 1)^{-1} (1 - \rho_i - \tau - \alpha)^{-1} \|A_1 v\|^{\Delta_i - 1} \times \right.$$

$$\times \|v\|^{2-\Delta_i} + \sum_{i=l+1}^k R_i \varepsilon_i^{-1} \|A_i^{\varepsilon_i} v\| + \sum_{i=k+1}^m R_i (1 - \Delta_i)^{-1} \|v\| \Big]. \quad (3)$$

Moreover, if $\tau < 0$ and $\alpha > 0$, then it is understood that

$$A_2^\tau (A_1^\alpha - A_2^\alpha) A_1^s = A_2^\tau A_1^{\alpha+s} - A_2^{\tau+\alpha} A_1^s.$$

Let $\max \rho_i < 1$ and $\theta = \max(\rho_i + \delta_i)$. Then it follows from (3) that, for any $\max(0, 1 - \theta) < \alpha < 1 - \max \rho_i$, $\varepsilon > 0$, the inequality

$$\|(A_1^\alpha - A_2^\alpha)v\| \leq cR\varepsilon^{-1} \|A_1^{\theta+\alpha+\varepsilon-1}v\| \quad (R = \max R_i) \quad (4)$$

holds.

* For the basic results on w.p. E -operators see ⁽¹⁻⁶⁾; some of their proofs see in ⁽⁷⁾. Perturbation theories in Hilbert space are devoted to ⁽⁸⁻¹⁰⁾.

Let $\theta \geq 1$; then it follows from this that $D(A_1^{\theta+\alpha+\varepsilon-1}) \subset D(A_2^\alpha)$ for every $\varepsilon > 0$. Let $\theta < 1$; then from (4) it follows that $D(A_1^\alpha) \subset D(A_2^\alpha)$ and the operator $A_1^\alpha - A_2^\alpha$ is subordinate to the operator $A_1^{\alpha-\delta}$, where $\delta = 1 - \theta - \varepsilon > 0$, since $\varepsilon > 0$ is arbitrary. Finally, if $\theta < 1$ and $0 \leq \alpha < 1 - \theta$, then

$$\|(A_1^\alpha - A_2^\alpha)v\| \leq cR(1 - \theta - \alpha)^{-1} \|v\|. \quad (5)$$

i.e., the operator $A_1^\alpha - A_2^\alpha$ admits closure to a bounded operator.

Let us consider the general example. Let A_1 and A_2 be s.p. E -operators, $D(A_1) = D(A_2)$, and suppose that for some $0 < \delta < 1$ the inequality

$$\|(A_1 - A_2)v\| \leq R \|A_i v\|^\delta \|v\|^{1-\delta} \quad (v \in D(A_i), \quad i = 1 \text{ or } i = 2) \quad (6)$$

holds.

Then $D(A_1^\alpha) = D(A_2^\alpha)$ for $0 \leq \alpha \leq 1$, and the inequalities

$$\|(A_1^\alpha - A_2^\alpha)v\| \leq cR\varepsilon^{-1} \|A_i^{\alpha+\delta+\varepsilon-1}v\| \quad (1 - \delta \leq \alpha \leq 1, \quad \varepsilon > 0),$$

$$\|(A_1^\alpha - A_2^\alpha)v\| \leq cR(1 - \delta - \alpha)^{-1} \|v\| \quad (0 \leq \alpha < 1 - \delta). \quad (7)$$

are valid.

3. The results of item 2 can be refined in the case when A_1 and A_2 are positive definite self-adjoint operators.

Suppose that (1) is fulfilled ($A_2^* = A_2$) and that $\Phi_i(v, w)$ satisfy the stronger condition than (2),

$$|\Phi_i(v, w)| \leq R_i \|A_1^{\delta_i} v\| \cdot \|A_2^{\rho_i} w\|. \quad (8)$$

Theorem 2. Suppose

$$-\min \delta_i \leq s \leq 1 - \max \delta_i, \quad -\min \rho_i < \tau < 1 - \max \rho_i,$$

$$-\min \rho_i - \tau < \alpha < 1 - \tau - \max \rho_i.$$

Then, for any $v \in D(A^\gamma)$, $\gamma = \max(1, \alpha + s)$, the inequality

$$\begin{aligned} \|A_2^\tau (A_1^\alpha - A_2^\alpha) A_1^s v\| &\leq c |\sin \alpha| \sum_{i=1}^m [(1 - \max \rho_i - \tau)(\tau + \min \rho_i) \\ &\quad \times (1 - \tau - \max \rho_i - \alpha)(\alpha + \tau + \min \rho_i)]^{1/2} \|A_1^{\Delta_i - 1} v\|. \end{aligned} \quad (9)$$

Here, if $\tau < 0$ and $\alpha > 0$, then it is assumed that

$$A_2^\tau (A_1^\alpha - A_2^\alpha) A_1^s \equiv A_2^\tau A_1^{\alpha+s} - A_2^{\tau+\alpha} A_1^s.$$

In particular, it follows from this that for $0 < \min \rho_i$, $\max \rho_i < 1$, $0 \leq \alpha < 1 - \max \rho_i$, and $\theta = \max(\rho_i + \delta_i)$, the inequality

$$\|(A_1^\alpha - A_2^\alpha) v\| \leq cR \|A_1^{\alpha+\theta-1} v\| \quad (R = \max R_i). \quad (10)$$

is valid.

It follows that $D(A_1^\alpha) \subset D(A_2^\alpha)$ for $\theta \leq 1$ and $0 \leq \alpha < 1 - \max \rho_i$. If $\theta < 1$, then the operator $A_1^\alpha - A_2^\alpha$ is subordinate to the operator $A_1^{\alpha-\delta}$, where $\delta = 1 - \theta > 0$. Therefore, for all $0 \leq \alpha \leq 1 - \theta$, the operator $A_1^\alpha - A_2^\alpha$ admits closure to a bounded operator.

Let us consider the general example. Let A_1 and A_2 be positive definite self-adjoint operators, $D(A_1) = D(A_2)$, and suppose that for some $0 \leq \delta \leq 1$ the inequality

$$\|(A_1 - A_2)v\| \leq R \|A_i^\delta v\|. \quad (11)$$

holds.

Then (cf. (8)) from Theorem 1 and (11) it follows that for any $0 \leq \alpha \leq 1$

$$\|(A_1^\alpha - A_2^\alpha)v\| \leq cR \|A_i^{\alpha+\delta-1} v\|. \quad (12)$$

Remark 1. It follows from Theorem 1 that all assertions of this item are valid for s.p. H -operators A_i , differing from self-adjoint ones by subordinate operators.

4. Let $A(t)$ be an operator-valued function defined on $[0, T]$ with values in the set of s.p. E of operators. Theorems 1 and 2 make it possible to estimate the smoothness of the operator-valued function $A^\alpha(t)$ in terms of the smoothness of the operator-valued function $A(t)$. We give only a theorem on the differentiability of $A^\alpha(t)$.

Theorem 3. Let the operator-valued function $[A(t) + \lambda I]^{-1}$ be strongly differentiable with respect to t on $[0, T]$, and let the form $\Phi(v, w) = -(A[t_1]v, w) - (v, A^*[t_2]w)$ and the numbers s and τ satisfy the conditions of item 2 for $m = 1$ and $R_1 = c|t_1 - t_2|$. Then, for every $-1 < \alpha < 1 - \rho_1$, the operator-valued function $A^\alpha(t)A^{-1}(0)$ is strongly differentiable with respect to t on $[0, T]$, and the inequalities

$$\|A^\tau(0)[A^\alpha(t)]'A^s(0)v\| \leq \begin{cases} c(\Delta_1 - 1)^{-1}(1 - \rho_1 - \tau - \alpha)^{-1}\|A(0)v\|^{\Delta_1-1}\|v\|^{2-\Delta_1} & (1 < \Delta_1 < 2), \\ c\varepsilon^{-1}\|A^\varepsilon(0)v\| & (\Delta_1 = 1, \varepsilon > 0), \\ c(1 - \Delta_1)^{-1}\|v\| & (\Delta_1 < 1). \end{cases} \quad (13)$$

hold.

If, however, $A(t)$ for each $t \in [0, T]$ is a positive definite self-adjoint operator, and the form $\Phi(v, w)$ and the numbers s and τ satisfy the conditions of item 3 for $m = 1$ and $R_1 = C|t_1 - t_2|$, then the inequality

$$\|A^\tau(0)[A^\alpha(t)]'A^s(0)v\| \leq c|\sin \pi \alpha| \times \quad (14)$$

$$\times [(1 - \rho_1 - \tau)(\tau + \rho_1)(1 - \tau - \rho_1 - \alpha)(\alpha + \tau + \rho_1)]^{1/2} \|A^{\Delta_1-1}(0)v\|$$

is valid.

Remark 2. The last assertion of the theorem is valid for s.p. H of operators $A(t)$ differing from a self-adjoint operator by a subordinate operator.

Remark 3. If the form $\Phi(v, w)$ satisfies the conditions of item 2 for $m = 1$, $\rho_1 + \delta_1 \leq 1$, and $R_1 \rightarrow 0$ as $t_1 - t_2 \rightarrow 0$, then for all $t \in [0, T]$ the inequality

$$\|[A(t) + \lambda I]^{-1}\| \leq C(1 + \lambda)^{-1}$$

is valid.

5. Let Ω be a domain of n -dimensional space with boundary S . Consider an operator A acting in $L_p(\Omega)$ ($1 < p < \infty$), defined by the elliptic differential expression

$$-\sum_{i,k=1}^n [a_i^k(x)v'_{x_k}]'_{x_i} + \sum_{i=1}^n a_i(x)v'_{x_i} + a(x)v \quad (x \in \Omega) \quad (15)$$

on functions from $W_{p,A}^{(2)}(\Omega)$ satisfying the boundary condition

$$-\sum_{i,k=1}^n a_{ik}(x)v'_{x_k} \cos(N, x_i) + \sigma(x)v = 0 \quad (x \in S). \quad (16)$$

By B we shall denote the operator generated by the boundary condition $v = 0$ ($x \in S$). If $a(x) \geq a_0$ and a_0 is sufficiently large, then A and B are s.p. $L_p(\Omega)$ -operators (see, for example, (12)). If the coefficients of (15) and (16) depend, respectively, on t and z , then we obtain operators $A(t, z)$ and $B(t)$.

Theorem 4. If $0 \leq \alpha < 1 - 1/2q$ ($1/q + 1/p = 1$), then

$$A^\alpha(t, z_1) - A^\alpha(t, z_2)$$

is subordinate to the operator

$$A^{\alpha-1/2+\varepsilon}(t, z_3)$$

($\varepsilon > 0$). If $p = 2$, then $\varepsilon = 0$ and $D[A^\alpha(t_1, z)] = D[B^\alpha(t_2)]$ for $0 \leq \alpha < 1/4$.

The proof uses the results from (13, 14). The equality

$$D[A^\alpha(t_i, z)] = D[B^\alpha(t_2)]$$

also follows from (15).

Theorem 3 makes it possible to prove the differentiability of $A^\alpha(t, z)A^{-1}(t_0, z_0)$ and $B^\alpha(t)B^{-1}(t_0)$, if the coefficients of (15) and (16) are differentiable with respect to t and z .

6. Consider semibounded elliptic operators A_1 and A_2 of order $2m$ with normal boundary conditions. If B is a differential operator of order $k < 2m$, then the operator $(A_i + B)^\alpha - A_i^\alpha$ is subordinate in $L_p(\Omega)$ to the operator $A_i^{\alpha+k/2m-1+\varepsilon}$ ($\varepsilon > 0$), where $\varepsilon = 0$ in the case of the self-adjoint operator A_i in $L_2(\Omega)$. If A_i are generated by the same

by a differential expression and boundary conditions with identical principal parts, then there exist such μ, ν in $(0, 1)$ that, for $0 \leq \alpha < \mu$, the operator $A_1^\alpha - A_2^\alpha$ is subordinate to the operator $A_i^{\alpha-\nu+\varepsilon}$ ($\varepsilon > 0$).

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REFERENCES

1. M. A. Krasnosel'skii, P. E. Sobolevskii, DAN, 129, No. 3 (1959).
2. P. E. Sobolevskii, UMN, 16, issue 4 (1961).

3. P. E. Sobolevskii, *Theory of Fractional Powers of Operators in a Banach Space and Its Applications to the Study of Equations of Parabolic Type*, Doctoral Dissertation, Moscow, 1961.
4. P. E. Sobolevskii, DAN, 166, No. 6 (1966).
5. A. V. Balakrishnan, Pacific J. Math., 10, No. 2 (1960).
6. T. Kato, Proc. Japan. Acad., 36, 94 (1960).
7. M. A. Krasnosel'skii, P. P. Zabreiko, E. I. Pustyl'nik, P. E. Sobolevskii, *Integral Operators in Spaces of Summable Functions*, "Nauka," 1966.
8. Yu. L. Daletskii, Tr. seminarov po funktsional'nomu analizu, vol. 6, Voronezh, 1958.
9. O. M. Kozlov, Tr. seminarov po funktsional'nomu analizu, vol. 6, Voronezh, 1958.
10. T. Kato, J. Math. Soc. Japan, 13, 246 (1961).
11. E. Heinz, Math. Ann., 123, 415 (1951).
12. P. E. Sobolevskii, Tr. Mosk. matem. obshch., 10, 297 (1961).
13. S. Agmon, A. Douglis, L. Nirenberg, Comm. Pure and Appl. Math., 12, 623 (1959).
14. V. P. Glushko, S. G. Krein, DAN, 122, No. 6 (1958).
15. S. G. Krein, Tr. IV Vsesoyuzn. matem. s"ezda, 1961, 2, "Nauka," 1964, p. 504.

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