

ON OPTIMAL PROCESSES IN TWO-PARAMETER DISCRETE SYSTEMS

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Abstract

Full Text

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MATHEMATICS

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ON OPTIMAL PROCESSES IN TWO-PARAMETER DISCRETE SYSTEMS

(Presented by Academician L. S. Pontryagin on 25 VIII 1966)

1. Let, at the nodes $(mh, n\tau)$ of the rectangle

$$D = \{mh, n\tau : m = 0, 1, \dots, M, n = 0, 1, \dots, N\},$$

the relation between the q -dimensional vector $x = x(m, n)$ and the r -dimensional control vector $u = u(m, n)$ be given by the equation

$$R_{h\tau}(x(m, n)) = f(x, R_h, R_\tau, u, m, n) \quad (1)$$

and by the initial conditions

$$x(0, n) = \gamma^n, \quad n = 0, 1, \dots, N; \quad x(m, 0) = \beta^m, \quad m = 0, 1, \dots, M, \quad (2)$$

where

$$\begin{aligned} R_{h\tau}(x(m, n)) &= [x(m+1, n+1) - x(m+1, n) - x(m, n+1) + \\ &\quad + x(m, n)]/h\tau, \\ R_h &= R_h(x(m, n)) = [x(m+1, n) - x(m, n)]/h, \\ R_\tau &= R_\tau(x(m, n)) = [x(m, n+1) - x(m, n)]/\tau, \end{aligned}$$

$f(x, R_h, R_\tau, u, m, n)$ is uniquely defined on

$$E^q \times E^q \times E^q \times E^r \times E^1 \times E^1,$$

is continuous, and has there continuous second derivatives with respect to x, R_h, R_τ .

Define the class of **admissible controls** ^(1,2) by requiring that $u(m, n) \in U$, where U is a given subset of E^r . On the trajectories (1)–(2) define the functional

$$S(u) = (c \cdot x(M, N)).$$

Problem. Find $u^0(m, n) \in U$ ($m = 0, 1, \dots, M-1$; $n = 0, 1, \dots, N-1$) such that

$$S(u^0) = \min_{u \in U} S(u). \quad (3)$$

A solution u^0 of problem (3) will be called an **optimal control**.

2. To the control $u(m, n) \in U$ there corresponds a solution $x(m, n)$; to the control

$$u(m, n) + \Delta u(m, n) \in U$$

there corresponds a solution $x(m, n) + \Delta x(m, n)$, and

$$\begin{aligned} R_{h\tau}(\Delta x(m, n)) &= f(x + \Delta x, R_h + \Delta R_h, R_\tau + \Delta R_\tau, u + \Delta u, m, n) \\ &\quad - f(x, R_h, R_\tau, u, m, n), \\ \Delta x(m, 0) &= \Delta x(0, n) = 0, \end{aligned} \quad (4)$$

where

$$\Delta R = R(x(m, n) + \Delta x(m, n)) - R(x(m, n)) = R(\Delta x(m, n)).$$

For an arbitrary sequence of q -dimensional vectors $p(m, n)$,

$$m = -1, 0, \dots, M-1; \quad n = -1, 0, \dots, N-1,$$

the following equality holds:

$$\begin{aligned} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} (p(m, n) \cdot R_{h\tau}(\Delta x(m, n))) h\tau &= \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} (\bar{R}_{h\tau}(p(m, n)) \cdot \Delta x(m, n)) h\tau \\ &\quad - \sum_{m=0}^{M-1} (\bar{R}_h(p(m, N-1)) \cdot \Delta x(m, N)) h \\ &\quad - \sum_{n=0}^{N-1} (\bar{R}_\tau(p(M-1, n)) \cdot \Delta x(M, n)) \tau \\ &\quad + (p(M-1, N-1) \cdot \Delta x(M, N)), \end{aligned} \quad (5)$$

where

$$\bar{R}_{h\tau}(p(m, n)) = [p(m, n) - p(m-1, n) - p(m, n-1) + p(m-1, n-1)]/h\tau,$$

$$\begin{aligned} \bar{R}_h(p(m, n)) &= [p(m, n) - p(m-1, n)]/h, \quad \bar{R}_\tau(p(m, n)) = [p(m, n) - \\ &\quad - p(m, n-1)]/\tau. \end{aligned}$$

Put

$$H(x, p, R_h, R_\tau, u, m, n) = (p(m, n) \cdot f(x, R_h, R_\tau, u, m, n))$$

and choose $p(m, n)$ from the equations

$$\begin{aligned} \bar{R}_{h\tau}(p(m, n)) &= \partial H(x, p, R_h, R_\tau, u, m, n) / \partial x - \\ &- \bar{R}_h(\partial H(x, p, R_h, R_\tau, u, m, n) / \partial R_h) - \bar{R}_\tau(\partial H(x, p, R_h, R_\tau, u, m, n) / \partial R_\tau) \end{aligned} \quad (6)$$

with boundary conditions

$$\begin{aligned} p(M-1, N-1) &= -c, \\ \bar{R}_h(p(m, N-1)) &= -\partial H(x, p, R_h, R_\tau, u, m, N-1) / \partial R_\tau, \\ \bar{R}_\tau(p, (M-1, n)) &= -\partial H(x, p, R_h, R_\tau, u, M-1, n) / \partial R_h. \end{aligned} \quad (7)$$

From (4), (6), (7) and identity (5) we obtain

$$\begin{aligned} -(c \cdot \Delta x(M, N)) &= \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} h\tau [H(x, p, R_h, R_\tau, u + \Delta u, m, n) - \\ &- H(x, p, R_h, R_\tau, u, m, n)] + \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} h\tau [H(x + \Delta x, p, R_h + \Delta R_h, R_\tau + \Delta R_\tau, \\ &u + \Delta u, m, n) - H(x, p, R_h, R_\tau, u + \Delta u, m, n) - \\ &- \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} h\tau \left[\left(\frac{\partial H(x, p, R_h, R_\tau, u, m, n)}{\partial x} \cdot \Delta x(m, n) \right) + \right. \\ &\left. + \left(\frac{\partial H(x, p, R_h, R_\tau, u, m, n)}{\partial R_h} \cdot \Delta R_h(m, n) \right) + \left(\frac{\partial H(x, p, R_h, R_\tau, u, m, n)}{\partial R_\tau} \cdot \Delta R_\tau(m, n) \right) \right]. \end{aligned}$$

Hence

$$\begin{aligned} \Delta S &\equiv S(u + \Delta u) - S(u) \equiv (c \cdot \Delta x(M, N)) = \\ &= - \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} h\tau [H(z, p, u + \Delta u, m, n) - H(z, p, u, m, n)] - \eta, \quad \eta + \eta_1 + \eta_2, \\ \eta_1 &= \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} h\tau \left(\left[\frac{\partial H(z, p, u + \Delta u, m, n)}{\partial z} - \frac{\partial H(z, p, u, m, n)}{\partial z} \right] \cdot \Delta z(m, n) \right), \\ \eta_2 &= \frac{1}{2} h\tau \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} (\Delta z(m, n) \times \end{aligned}$$

$$\times \frac{\partial^2 H(z + \theta \Delta z, p, u + \Delta u, m, n)}{\partial z^2} \Delta z(m, n) \Big), \quad 0 \leq \theta(m, n) \leq 1.$$

Here $z = \{x, R_h, R_\tau\}$ is a $3q$ -dimensional vector.

For the variation

$$\Delta^* u(m, n) = \begin{cases} u^* - u(k, l), & m = k, n = l, \\ 0, & m \neq k, n \neq l, \end{cases}$$

we have

$$\Delta^* S \equiv S(u + \Delta^* u) - S(u) = -h\tau \{ [H(z, p, u^*, k, l) - H(z, p, u, k, l)] - \eta^* \},$$

$$\eta^* = \frac{1}{2} \sum_{m=k+1}^{M-1} \sum_{n=l+1}^{N-1} \left(\Delta z(m, n) \cdot \frac{\partial^2 H(z + \theta \Delta z, p, u, m, n)}{\partial z^2} \Delta z(m, n) \right), \quad \eta_1^* = 0.$$

It is not difficult to show that

$$\begin{aligned} |\eta^*| &\leq \frac{1}{2} (h\tau)^2 \|f(z, u^*, k, l) - f(z, u, k, l)\|^2 \times \\ &\times \sum_{m=k+1}^{M-1} \sum_{n=l+1}^{N-1} L_{m,n} \left\| \frac{\partial^2 H(m, n)}{\partial z^2} \right\| = \bar{\eta}. \end{aligned} \quad (8)$$

3. Let $f(z, u, m, n)$ be differentiable with respect to u . Introduce the set $a(\alpha, k, l)$:

$$a(\alpha, z, p, k, l, u(k, l), \dots, u(M-1, N-1), h, \tau) = \{u^* : |\eta^*| \leq \alpha\}$$

and the quantity $\delta_u H$

$$H(z, p, u + \Delta u, m, n) - H(z, p, u, m, n) = \delta_u H(z, p, u, m, n) + o(\|\delta u\|).$$

Definition. The control u at the node (k, l) satisfies the quasimaximum condition with number $\mu(k, l)$ and set $\omega(k, l)$, if:

- 1) $H(z, p, u, k, l) \geq H(z, p, u^*, k, l) - \mu(k, l)$ for all $u^* \in \omega(k, l)$;
- 2) $\delta_u H \leq 0$ if U is convex,
- 3) $\delta_u H = 0$ at interior points of U .

Theorem 1. *The optimal control $u^0 = u^0(m, n)$ for problem (3) satisfies the conditions:*

- 1) *quasimaximum with $\mu(k, l) = \alpha$, $\omega(k, l) = a(\alpha, k, l) \cap U$, $k = 0, 1, \dots, M - 2$; $l = 0, 1, \dots, N - 2$;*
- 2)

$$H(z, p, u^0, k, l) \geq H(z, p, u, k, l), \quad u \in U \quad (9)$$

for $k = M - 1$; $l = 0, \dots, N - 1$ and $k = 0, \dots, M - 1$; $l = N - 1$.

Remark. For convenience of computation, the set $a(\alpha, k, l)$ may be replaced by the set $a'(\alpha, k, l)$:

$$a'(\alpha, k, l) = \{u^* : |\bar{\eta}| \leq \alpha\},$$

where $\bar{\eta}$ is some upper estimate for η^* , for example (8).

From the definition of the set $a(\alpha, k, l)$ it follows that the local-maximum principle is valid at the node (k, l) if the set $a(0, k, l)$ contains points distinct from $u^0(k, l)$.

Thus, the quasimaximum condition singles out some set $U^* \subset U$ of points $u(k, l)$ suspected of optimality. The narrowing of the set U^* is carried out by varying α and by the local conditions $\delta_u H \leq 0$, $\delta_u H = 0$.

Let $h \rightarrow 0$, $\tau \rightarrow 0$; then the number α^* from $a'(\alpha^*, k, l) \supseteq U$ tends to zero, and the quasimaximum condition passes into the condition of the absolute maximum, with respect to u , of the function H .

4. For the linear variant

$$\begin{aligned} R_{h\tau}(x(m, n)) &= A(m, n)x(m, n) + B(m, n)R_h(x(m, n)) + \\ &+ C(m, n)R_\tau(x(m, n)) + \varphi(m, n, u(m, n)) \end{aligned} \quad (10)$$

of equation (1), we have $\eta = 0$.

Theorem 2. *In order that the control $u^0 \in U$ for (10) be optimal, it is necessary and sufficient that condition (9) be fulfilled at the nodes (m, n) , $m = 0, 1, \dots, M - 1$; $n = 0, 1, \dots, N - 1$.*

5. For the case

$$\begin{aligned} R_{h\tau}(x(m, n)) &= A(m, n, u(m, n))x(m, n) + B(m, n, u(m, n))R_h(x(m, n)) + \\ &+ C(m, n, u(m, n))R_\tau(x(m, n)) + \varphi(m, n, u(m, n)) \end{aligned}$$

it is not difficult to see that $\eta^* = 0$.

Theorem 3. *If $u^0 = u^0(m, n)$ is an optimal control, then at each node (k, l) , $k = 0, 1, \dots, M - 1$; $l = 0, 1, \dots, N - 1$, (9) is fulfilled.*

6. Let U be convex and let equation (1) have the form

$$R_{h\tau}(x(m, n)) = \sum_{i=1}^r u_i(m, n) \cdot f_i(x, R_h, R_\tau, m, n) + g(x, R_h, R_\tau, m, n).$$

Theorem 4. *For the optimal control u^0 , condition (9) is fulfilled at all nodes of the rectangle D .*

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CITED LITERATURE

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