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Abstract

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MATHEMATICS

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A LIMIT THEOREM FOR THE NUMBER OF LEVEL CROSSINGS BY A STATIONARY GAUSSIAN PROCESS

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Consider a continuous with probability 1 stationary Gaussian process ξ_t with correlation function r_t . Let ξ_t be differentiable in mean square (m.s.), and let $N(s, t)$ denote the number of crossings from below to above of the level u by the process ξ_t on $[s, t]$. From ⁽¹⁾ it follows that $EN(s, t) < \infty$.

Lemma 1. *If $r_t \rightarrow 0$ as $t \rightarrow \infty$, then*

$$DN(0, T) \xrightarrow{T \rightarrow \infty} \infty$$

for all u .

Proof. Let $DN(0, T) < \infty$, otherwise the assertion of the lemma is trivial. Put

$$X_k(\tau) = N(k\tau, (k+1)\tau) - EN(k\tau, (k+1)\tau), \quad k = 0, 1, \dots \quad (1)$$

For any τ , $\{X_k(\tau)\}$ is a stationary sequence, and, by Theorem 2 on p. 119 in ⁽²⁾, $EX_0X_k \rightarrow 0$ as $k \rightarrow \infty$. Choose τ so that

$$EN(0, \tau) = 1/2. \quad (2)$$

Let $\varliminf_n DN(0, n\tau) < \infty$. Then, by Theorem 18.2.2 in ⁽³⁾, there is a representation:

$$X_k = Y_{k+1} - Y_k,$$

where $Y_k = U^k Y_0$, $Y_0 \in H(X)$, and $H(X)$ is the closure, in the m.s. sense, of the linear span of the set $\{X_0, X_1, \dots\}$, while U is a unitary operator in $H(X)$ defining X_k : $X_k = U^k X_0$. From (2) it follows that

$$\mathbf{P}\{|Y_{2n+1} - Y_0| < 1/2\} = 0. \quad (3)$$

Let $I = (a, a + 1/2)$ be an interval such that $\mathbf{P}\{Y_0 \in I\} = p \neq 0$. The variables Y_n , obviously, are measurable with respect to ξ_t , $0 \leq t < \infty$; therefore, by the mentioned Theorem 2 in ⁽²⁾,

$$\mathbf{P}\{Y_0, Y_n \in I\} \xrightarrow{n \rightarrow \infty} p^2,$$

which contradicts (3). Thus, $\lim_n DN(0, n\tau) = \infty$. Let $T \rightarrow \infty$, and put $n = [T/\tau]$. We have

$$DN(0, T) > DN(0, n\tau) - 2\sqrt{DN(0, n\tau) E(N(0, \tau))^2} \rightarrow \infty.$$

Lemma 2. *If ξ_t is twice differentiable in the m.s. sense and*

$$\int_0^\infty t(|r_t| + |r'_t| + |r''_t|) dt < \infty,$$

then, as $T \rightarrow \infty$,

$$DN(0, T) = \sigma^2 T(1 + o(1))$$

and $\sigma \neq 0$.

Proof. From ⁽⁵⁾ it follows that $DN(0, T) < \infty$. Let τ be fixed.

$$DN(0, n\tau) = nEX_0^2(\tau) + 2 \sum_{k=1}^{n-1} (n-k)EX_0(\tau)X_k(\tau), \quad (4)$$

where $X_k(\tau)$ is defined from (1). In the same way as in (4) when finding the factorial moments of $N(0, T)$, one can show that

$$\mathbf{E}X_0(\tau)X_k(\tau) = \int_0^\tau \left(\int_{k\tau}^{(k+1)\tau} (F_{t-s} - F_\infty) dt \right) ds, \quad k > 0, \quad (5)$$

where

$$F_{t-s} = \int_0^\infty \int_0^\infty y_1 y_2 \varphi_{t-s}(u, u, y_1, y_2) dy_1 dy_2, \quad F_\infty = \lim_{t \rightarrow \infty} F_t,$$

and $\varphi_{t-s}(x_1, x_2, y_1, y_2)$ is the density of the vector $\{\xi_s, \xi_t, \dot{\xi}_s, \dot{\xi}_t\}$. (We note that, by the conditions of the lemma, $r_t, r'_t, r''_t \rightarrow 0$ as $t \rightarrow \infty$, so the definition of F_∞ makes sense.) Let R_{t-s} and R_∞ be the correlation matrices corresponding to φ_{t-s} and $\varphi_\infty = \lim_{t \rightarrow \infty} \varphi_t$. Put $\delta_{t-s} = \|R_{t-s} - R_\infty\|$, where $\|(a_{ij})\| = \max_{ij} |a_{ij}|$. It is easy to obtain the estimate

$$\begin{aligned} & |\varphi_{t-s}(u, u, y_1, y_2) - \varphi_\infty(u, u, y_1, y_2)| \\ & < C \exp \left\{ -(4r_0 - 4r''_0)^{-1} (y_1^2 + y_2^2) \delta_{t-s} |R_{t-s}|^{-3/2} (1 + y_1^2 + y_2^2) \right\}, \end{aligned} \quad (6)$$

where C depends only on u, r_0, r''_0 . From (5) and (6) we obtain

$$\begin{aligned} & \sum_1^\infty |\mathbf{E} X_0 X_k| < 2 \int_0^\infty t |F_t - F_\infty| dt < \\ & < 2a \int_0^a |F_t - F_\infty| dt + 2C_1 \int_a^\infty t \delta_t |R_t|^{-3/2} dt. \end{aligned} \quad (7)$$

The first integral is finite, since $DN(0, a) < \infty$. It is obvious that $|R_t|$ is bounded below uniformly in t on $[a, \infty)$ for any $a > 0$. In addition, $\delta_t < |r_t| + |r'_t| + |r''_t|$; therefore the second integral on the right-hand side of (7) is also finite. Finally we obtain

$$\left| \sum_1^\infty \mathbf{E} X_0(\tau) X_k(\tau) \right| < K,$$

where K does not depend on τ . By Lemma 1, $\mathbf{E} X_0^2(\tau) \rightarrow \infty$ as $\tau \rightarrow \infty$. Choose τ so that $\mathbf{E} X_0^2(\tau) > 2K$. Now (4) can be rewritten as follows:

$$\begin{aligned} DN(0, n\tau) &= n \left(\mathbf{E} X_0^2(\tau) + 2 \sum_1^\infty \mathbf{E} X_0(\tau) X_k(\tau) + o(1) \right) = \\ &= n\sigma_1^2(1 + o(1)), \quad \sigma_1 \neq 0. \end{aligned}$$

Put $\sigma^2 = \sigma_1^2/\tau$, and let $T \rightarrow \infty$, $n = [T/\tau]$.

$$\begin{aligned} & \left| \frac{DN(0, T)}{\sigma^2 T} - 1 \right| < \frac{|DN(0, T) - DN(0, n\tau)|}{\sigma^2 T} + \\ & + DN(0, n\tau) \left(\frac{1}{\sigma^2 n\tau} - \frac{1}{\sigma^2 T} \right) + \left| \frac{DN(0, n\tau)}{\sigma^2 n\tau} - 1 \right| = o(1). \end{aligned}$$

Lemma 3. Let R_t denote the correlation matrix of the vector $\{\xi_0, \xi_t, \dot{\xi}_0, \dot{\xi}_t\}$. If $r_t^{(8)}$ is continuous, then in some neighborhood of zero

$$|R_t| = ct^8 + o(t^8), \quad c > 0.$$

Proof. Putting $r_0 = 1$, we find

$$|R_t| = (r_t^2 + r_0'' - r_t r_t'')^2 - (r_t' - r_t r_0'')^2.$$

It is enough to expand the terms in the parentheses up to t^6 , since the expansion will begin with t^2 . We obtain

$$|R_t| = \frac{t^8}{144} (r_0^{(4)} - r_0''^2) (r_0^{(6)} r_0'' - r_0^{(4)2}) + o(t^8).$$

From the positive definiteness of R_t it follows that the coefficient of t^8 is nonnegative. Let $r_0^{(4)} = r_0''/2$. This means that $E\xi_0^2 E\dot{\xi}_0^2 = (E\xi_0 \dot{\xi}_0)^2$. But in the range space of ξ_t , which is the closure in mean square of the linear span of the variables ξ_{t_k} , such an equality is possible only under the condition $E(\dot{\xi}_0 - \lambda \xi_0)^2 = 0$, where λ is some number; this cannot occur when the process ξ_t is Gaussian. Similarly we verify that $r_0^{(6)} r_0'' - r_0^{(4)2} > 0$.

Lemma 4. If there exists and is continuous in mean square $d^4 \xi_t / dt^4$, and $r_t, r_t', r_t'' = O(t^{-\alpha})$ as $t \rightarrow \infty$ for some $\alpha > 13$, then for all u

$$\mu_4(T) = E(N(0, T) - EN(0, T))^4 = O(T^2) \quad \text{as } T \rightarrow \infty.$$

Proof. From (5) it follows that $\mu_4(T) < \infty$. In the notation (1) we have

$$\begin{aligned} \mu_4(n\tau) &< nEX_0^2 + \sum_{i \neq j} |E(X_i^2 X_j^2 + X_i^3 X_j)| + \sum_{i \neq j \neq k} |EX_i^2 X_j X_k| \\ &+ \sum_{i \neq j \neq k \neq l} |EX_i X_j X_k X_l| \\ &= O \left(n^2 + \sum_{i < j-1 < k-2} |EX_i^2 X_j X_k| + \sum_{i < j-1 < k-2 < l-3} |EX_i X_j X_k X_l| \right). \end{aligned} \quad (8)$$

Put $N(i\tau, (i+1)\tau) = N_i$, $N_i(N_i - 1) = N_i^{(2)}$. Consider the second term on the right-hand side of (8):

$$EX_i^2 X_j X_k = E(N_i^{(2)} N_j + N_i N_j - (N_i^{(2)} + N_i) EN_j - (EN_i)^2 N_j) X_k. \quad (9)$$

As in (4), we obtain:

$$\mathbb{E} N_i^{(2)} N_j X_k = \int_{\Delta_i \times \Delta_i} dt_1 dt_2 \int_{\Delta_j} dt_3 \int_{\Delta_k} (F_{t_1 t_2 t_3 t_4} - F_{t_1 t_2 t_3 \infty}) dt_4, \quad (10)$$

where

$$F_{t_1 t_2 t_3 t_4} = \int_0^\infty \cdots \int_0^\infty y_1 \cdots y_4 \varphi_{t_1 \dots t_4}(u, \dots, u, y_1, \dots, y_4) dy_1 \cdots dy_4$$

and $\varphi_{t_1 \dots t_4}(x_1, \dots, x_4, y_1, \dots, y_4)$ is the density of the vector $\{\xi_{t_1}, \dots, \xi_{t_4}, \dot{\xi}_{t_1}, \dots, \dot{\xi}_{t_4}\}$, $F_{t_1 t_2 t_3 \infty} = \lim_{t_4 \rightarrow \infty} F_{t_1 t_2 t_3 t_4}$, with $t_1 \neq t_2 \in \Delta_i = (i\tau, (i+1)\tau)$, $t_3 \in \Delta_j$. Denote by $R_{t_2-t_1}$ the correlation matrix of the vector $\{\xi_{t_1}, \xi_{t_2}, \dot{\xi}_{t_1}, \dot{\xi}_{t_2}\}$. By Lemma 3 there exists t_0 such that for all t_1, t_2 for which $|t_1 - t_2| < t_0$,

$$|R_{t_2-t_1}| > c(t_2 - t_1)^8, \quad c > 0.$$

Introduce the function

$$\beta_k = \beta_k(\tau) = \max_{s > k\tau} (|r_s| + |r'_s| + |r''_s|).$$

Put

$$A_0 = \{t_1, t_2 \in \Delta_i : |t_1 - t_2| < \beta_{j-i-1}^{1/13} + \beta_{k-j-1}^{1/13}\},$$

$$A_1 = \{t_1, t_2 \in \Delta_i : |t_1 - t_2| \geq t_0\}, \quad A_2 = (\Delta_i \times \Delta_i) \setminus (A_0 \cup A_1).$$

For sufficiently large τ , $A_0 A_1 = \emptyset$ for all $i < j-1 < k-2$. We split the outer integral in (10) into three, over the sets A_0, A_1, A_2 , respectively. The first integral will be of order $\beta_{j-i-1}^{1/13} + \beta_{k-j-1}^{1/13}$, as follows from the proof of Theorem 3 in (5). The second is estimated in the same way as in the proof of Lemma 2. To estimate the integral over A_2 , we use the obvious analogue of relation (6). We obtain:

$$|F_{t_1 \dots t_4} - F_{t_1 t_2 t_3 \infty}| < C \delta_{t_1 \dots t_4} [\min\{|R_{t_1 \dots t_4}|, |R_{t_1 t_2 t_3 \infty}|\}]^{-3/2},$$

where C depends only on u, r_0, r'_0 . Obviously, on A_2

$$\min\{|R_{t_1 \dots t_4}|, |R_{t_1 \dots \infty}|\} > c_1(\beta_{j-i-1}^{1/13} + \beta_{k-j-1}^{1/13}),$$

and, since $\delta_{t_1 \dots t_4} = |R_{t_1 \dots t_4} - R_{t_1 \dots \infty}| < \beta_{k-j-1}$, it follows that

$$|F_{t_1 \dots t_4} - F_{t_1 \dots \infty}| < C_1(\beta_{j-i-1}^{1/13} + \beta_{k-j-1}^{1/13}).$$

Thus, for the integral (10) there is the estimate

$$|EN_i^{(2)} N_j X_k| < C_2(\beta_{j-i-1}^{1/13} + \beta_{k-j-1}^{1/13}).$$

Similarly one estimates $|EN_i^{(2)} X_k|$. For the other terms in (10) we easily obtain $|EN_i N_j X_k| < C_3 \beta_{k-j-1}$, $|EN_j^2 X_k| < C_4 \beta_{k-j-1}$. Finally,

$$\sum_{i < j-1 < k-2} |EX_i^2 X_j X_k| = O\left(\sum_{i < j-1 < k-2} (\beta_{j-i-1}^{1/3} + \beta_{k-j-1}^{1/3}) + n^2\right) = O(n^2).$$

For the last term in (8) we find the estimate:

$$\begin{aligned} & \sum_{i < j-1 < k-2 < l-3} |EX_i X_j X_k X_l| = \\ & = O\left(\sum_{i < j-1 < k-2 < l-3} \min(\beta_{j-i-1}, \beta_{l-k-1})\right) = O\left(n^2 \sum_0^\infty i \beta_i\right) = O(n^2). \end{aligned}$$

Thus, $\mu_4(n\tau) = O(n^2)$ for some sufficiently large τ . Hence the assertion of Lemma 4 follows.

Suppose now that ξ_t satisfies the strong mixing condition in the sense that

$$\rho(t) = \sup_{\substack{A \in \mathcal{F}_{-\infty}^0 \\ B \in \mathcal{F}_t^\infty}} |\mathbf{P}(AB) - \mathbf{P}(A)\mathbf{P}(B)| \xrightarrow[t \rightarrow \infty]{} 0, \quad (11)$$

where $\mathcal{F}_a^b$ is the σ -algebra generated by ξ_t , $t \in [a, b]$. As a consequence of Lemmas 2 and 4, the following is obtained.

Theorem. *Under the conditions of Lemma 4 and (11)*

$$\mathbf{P}\left\{\frac{N(0, T) - EN(0, T)}{\sqrt{DN(0, T)}} < z\right\} \rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^z e^{-x^2/2} dx \quad (12)$$

as $T \rightarrow \infty$, for all u .

Proof. According to Theorem 18.4.1 in (3), in order to prove (12) for $T = n\tau$, where τ is fixed, it is sufficient to verify, for any integer-valued function $p = p(n)$ such that $p \rightarrow \infty$, $p = o(n)$, and any $\varepsilon > 0$, the relation

$$\frac{n}{p\sigma_n^2} \int_{|z| > \varepsilon \sigma_n} z^2 dF_p(z) \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

where $\sigma_n^2 = DN(0, n\tau)$, $F_p(z) = \mathbf{P}\{N(0, p\tau) - EN(0, p\tau) < z\}$. We have

$$\frac{n}{p\sigma_n^2} \int_{|z| > \varepsilon \sigma_n} z^2 dF_p(z) < \frac{n\mu_4(p\tau)}{\varepsilon^2 p\sigma_n^4} = O\left(\frac{p}{n}\right) = o(1).$$

Now, for arbitrary T , (12) is proved with the aid of the obvious proposition: if $\mathbf{P}\{\xi_n + \eta_n < x\} \rightarrow F(x)$, where F is some distribution function and $\eta_n \rightarrow 0$ in probability, then $\mathbf{P}\{\xi_n < x\} \rightarrow F(x)$.

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