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Abstract

Full Text

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Aerodynamics

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ON THE CONSTRUCTION OF SOLUTIONS OF THE SYSTEM OF EQUATIONS OF BOUNDARY-LAYER THEORY BY THE METHOD OF LINES

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The question of the existence and uniqueness of solutions of the Prandtl system for two-dimensional stationary and nonstationary flows of an incompressible fluid was studied in works ^(1,2). Below, solutions of the Prandtl system are constructed for axisymmetric three-dimensional nonstationary flows and two-dimensional flows by means of the method of lines, which is a variant of the well-known method of integral relations of A. A. Dorodnitsyn ^(3,4).

1. The Prandtl system for axisymmetric three-dimensional nonstationary flows of an incompressible fluid in the corresponding coordinates has the form (see ⁽⁵⁾, p. 174)

$$u_t + uu_x + vu_y = -p_x + \nu u_{yy}, \quad (ru)_x + (rv)_y = 0 \quad (1)$$

in the domain $D\{0 \leq t \leq t_0, 0 \leq x \leq x_0, 0 \leq y < \infty\}$,

$$u|_{t=0} = u_0(x, y), \quad u|_{x=0} = 0, \quad u|_{y=0} = 0, \quad v|_{y=0} = v_0(t, x),$$

$$u \rightarrow U(t, x) \quad \text{as } y \rightarrow \infty. \quad (2)$$

The function $r(x)$ determines the surface of the body being flowed around, $r(0) = 0$, $r(x) > 0$ for $x > 0$. By Bernoulli's law, $-p_x = U_t + UU_x$; $U(t, 0) = 0$, $U(t, x) > 0$ for $x > 0$.

If one introduces new independent variables

$$\eta = u(t, x, y)/U(t, x), \quad \xi = x, \quad \tau = t, \quad (3)$$

then for the function $w = u_y/U$ we obtain the equation

$$\nu w^2 w_{\eta\eta} - w_\tau - \eta U w_\xi + A w_\eta + B w = 0 \quad (4)$$

in the domain $\Omega\{0 \leq \tau \leq t_0, 0 \leq \xi \leq x_0, 0 \leq \eta \leq 1\}$ with the conditions

$$w|_{t=0} = u_{0y} = w_0(\xi, \eta), \quad w|_{\eta=1} = 0, \quad (\nu w w_\eta - v_0 w + C)|_{\eta=0} = 0, \quad (5)$$

where

$$A \equiv (\eta^2 - 1)U_x + (\eta - 1)U_t U^{-1}, \quad B \equiv \eta r' r^{-1}U - \eta U_x - U_t U^{-1},$$

$$C \equiv U_x + U_t U^{-1}.$$

Using the method of lines, we shall prove, under appropriate assumptions, the existence and uniqueness of the solution of problem (4), (5), and then, as a consequence, obtain theorems on the existence and uniqueness of the solution of the Prandtl system (1), (2).

Let $f^{m,l} \equiv f(mh, lh, \eta)$. We replace equation (4) with conditions (5) by a system of ordinary differential equations

$$\begin{aligned} & \nu(w^{m-1,l} + h)^2 w_{\eta\eta}^{m,l} - (w^{m,l} - w^{m-1,l})/h \\ & - (\eta + h)U^{m,l}(w^{m,l} - w^{m,l-1})/h + A^{m,l}w_\eta^{m,l} + B^{m,l}w^{m,l} = 0 \end{aligned} \quad (6)$$

with the conditions

$$w^{m,l}(1) = 0, \quad \nu w^{m-1,l}(0)w_\eta^{m,l}(0) - v_0^{m,l}w^{m-1,l}(0) + C^{m,l} = 0, \quad (7)$$

$$m = 1, 2, \dots, \quad l = 0, 1, 2, \dots; \quad w^{0,l} \equiv w_0(lh, \eta), \quad h = \text{const} > 0.$$

The solution of (6), (7) is reduced to the successive solution of linear second-order equations with prescribed boundary conditions for $m = 1, l = 0, 1, 2, \dots$, then $m = 2, l = 0, 1, 2, \dots$, etc. The solution of problem (6), (7) obviously exists if $w^{m-1,l}(0) \neq 0$ and $w^{m-1,l}(\eta) \geq 0$.

Lemma 1. *Let A, B, C, v_0 be bounded functions in Ω , let w_0 be continuous, and let $K_1(1 - \eta) \leq w_0(\xi, \eta) \leq K_2(1 - \eta)$, $K_1 > 0$, $K_2 > 0$, be constants. Then the solution of (6), (7) exists for $mh \leq \tau_0$, where τ_0 is a certain number depending on the data of problem (1)-(2), and the estimate*

$$V(mh, \eta) \leq w^{m,l}(\eta) \leq V_1(mh, \eta)$$

holds, where the functions V and V_1 , continuous in Ω , are positive for $\eta < 1$, $V = K_3(1 - \eta)$, $V_1 = K_4(1 - \eta)$ in a neighborhood of $\eta = 1$, and $K_3 > 0$, $K_4 > 0$ are constants.

Lemma 2. Let the conditions of Lemma 1 be satisfied; let A, B, C, v_0, w_0 have bounded first derivatives,

$$|w_{0\xi}| \leq K_5(1 - \eta),$$

$w_0 w_{0\eta\eta}$ be bounded in Ω , and let the compatibility condition

$$\nu w_0 w_{0\eta} - v_0 w_0 + C = 0$$

hold for $\tau = 0$, $\eta = 0$. Then

$$w_\eta^{m,l}, \quad (w_\eta^{m,l} - w_\eta^{m-1,l})/h, \quad (w_\eta^{m,l} - w_\eta^{m,l-1})/h, \quad (1 - \eta + h)w_\eta^{m,l}$$

are bounded in Ω for $mh < \tau_1$ by constants independent of h ; τ_1 is a certain number determined by the data of problem (1)-(2).

Theorem 1. Let the assumptions of Lemmas 1 and 2 be fulfilled. Then in the domain Ω , for $\tau \leq \tau_1$, there exists a solution w of problem (4), (5) possessing the following properties: w is continuous in Ω and

$$K_6(1 - \eta) \leq w \leq K_7(1 - \eta),$$

$K_i = \text{const} > 0$; w has bounded generalized derivatives w_η , w_τ , w_ξ ; the derivative w_η is continuous with respect to η for $\eta < 1$, w satisfies the conditions (5); there exists a generalized derivative $w_{\eta\eta}$ such that $(1 - \eta)w_{\eta\eta}$ is bounded in Ω ; the function w satisfies equation (4) almost everywhere in Ω . The solution w of problem (4), (5) possessing the indicated properties is unique.

The uniqueness of the solution w of problem (4), (5) follows from the energy inequality that holds for the equation satisfied by the difference of two solutions.

The functions $w_\eta^{m,l}(\eta)$, correspondingly extended linearly in τ and ξ in Ω , according to Lemmas 1, 2 form a compact family for $\tau \leq \tau_1$. Some subsequence of these functions converges uniformly to a function w as $h \rightarrow 0$. From the uniqueness of the limiting function w it follows that the whole family $w_\eta^{m,l}$ converges as $h \rightarrow 0$.

Theorem 2. Let $U_x, U_t U^{-1}, r' r^{-1} U, v_0$ have bounded derivatives with respect to t and x in D , $u_0 \rightarrow U$ as $y \rightarrow \infty$, $u_{0y} > 0$,

$$M_1(U - u_0) \leq u_{0y} \leq M_2(U - u_0),$$

$M_1 > 0$, $M_2 > 0$ constants; let there exist bounded derivatives u_{0yy} , u_{0yyy} , u_{0x} , u_{0xy} , let the quantities u_0/U , u_{0yy}/u_{0y} , and

$$(u_{0yyy}u_{0y} - u_{0yy}^2)/u_{0y}^2$$

be bounded and, moreover,

$$|(u_{0yx} - u_{0x}u_{0yy}/u_{0y}) + U_x U^{-1}(u_0 u_{0yy} - u_{0y}^2)/u_{0y}| \leq M_3(U - u_0).$$

Let the compatibility condition

$$v_0(0, x)u_{0y}(x, 0) = -p_x(0, x) + \nu u_{0yy}(x, 0) \quad (8)$$

be fulfilled. Then in D there exists, uniquely for $t \leq \tau_1$, a solution u, v of problem (1), (2) possessing the following properties: u, u_y are continuous and bounded, $u_y > 0$ for $y \geq 0$, $u_y \rightarrow 0$ as $y \rightarrow \infty$, v is continuous in y and bounded for bounded y . There exist bounded generalized derivatives u_t, u_x, u_{yy}, v_y , and the system (1) is satisfied almost everywhere; in addition, the following quantities are bounded:

$$u_y/U, \quad u_{yy}/u_y, \quad (u_{yyy}u_y - u_{yy}^2)/u_y^2, \\ U^{-1}(u_{yx} - u_x u_{yy}/u_y) + U_x(uu_{yy} - u_y^2)/u_y U^2, \quad U^{-1}(u_{yt} - u_t u_{yy}/u_y) + U_t(uu_{yy} - u_y^2)/u_y U^2.$$

In order to obtain a smoother solution of problem (1), (2), we must consider another system of ordinary differential equations approximating (4), (5). By $f^{m,l,n}$ we shall denote

value of the function $f_n(\tau, \xi, \eta)$ at the point (mh, lh, η) . For $0 \leq \eta \leq 1$ consider the system

$$\nu (w^{m,l,n-1})^2 w_{\eta\eta}^{m,l,n} - (w^{m,l,n} - w^{m-1,l,n})/h \\ - (\eta + h)U^{m,l} (w^{m,l,n} - w^{m,l-1,n})/h + A^{m,l} w_{\eta}^{m,l,n} + B^{m,l} w^{m,l,n} = 0 \quad (9)$$

with the conditions

$$w^{m,l,n}(1) = 0, \quad \nu w^{m,l,n-1}(0)w_{\eta}^{m,l,n}(0) - v_0^{m,l}w^{m,l,n-1}(0) + C^{m,l} = 0, \quad (10)$$

$$n, m = 1, 2, \dots, \quad l = 0, 1, 2, \dots, \quad w^{0,l,n} = w_0(lh, \eta), \\ w^{m,l,0} = w_0(lh, \eta).$$

To prove the existence of a solution of (9), (10) with continuous derivatives of fourth order for $0 \leq \eta \leq 1$ for $mh \leq \tau_2$, we use the method of introducing a small parameter and enlarging the domain, as was done in constructing smooth solutions in papers ^(2, 6).

Lemma 3. Let the conditions of Lemmas 1 and 2 be satisfied; A, B, C, v_0, w have bounded derivatives of second order; $w_0^2 w_{0\eta\xi}, w_0^2 w_{0\eta\eta\eta}, w_0^2 w_{0\eta\eta\eta\eta}$ are bounded, and suppose the compatibility condition

$$\nu w_0 w_{\tau} + \nu w_{0\eta} w_{\tau} - v_{0\tau} w_0 - v_0 w_{\tau} + C_{\tau} = 0$$

is satisfied for $\eta = 0, \tau = 0$, where

$$w_{\tau} \equiv \nu w_0^2 w_{0\eta\eta} + A w_{0\eta} + B w_0.$$

Then in Ω , for $\tau \leq \tau_3$, the quantities

$$\begin{aligned} & w^{m,l,n}, \quad w_{\eta}^{m,l,n}, \quad (w^{m,l,n} - w^{m-1,l,n})/h, \\ & (w^{m,l,n} - w^{m,l-1,n})/h, \quad w_{\eta\eta}^{m,l,n}, \quad (w^{m+1,l,n} - 2w^{m,l,n} + w^{m-1,l,n})/h^2, \\ & (w_{\eta}^{m,l,n} - w_{\eta}^{m-1,l,n})/h, \quad (w_{\eta}^{m,l,n} - w_{\eta}^{m,l-1,n})/h, \\ & (w^{m,l+1,n} - 2w^{m,l,n} + w^{m,l-1,n})/h^2, \\ & (w^{m,l,n} - w^{m,l-1,n} - w^{m-1,l,n} + w^{m-1,l-1,n})/h^2 \end{aligned}$$

are bounded by constants independent of n and h ; $\tau_3 > 0$ and is determined by the data of problem (1), (2).

It follows from Lemma 3 that the solution w of problem (4), (5), whose existence is asserted in Theorem 1, has, for $\tau < \tau_3$, under the assumptions of Lemma 3, first-order derivatives satisfying the Lipschitz condition and a bounded derivative $w_{\eta\eta}$, continuous for $\eta < 1$. Hence we obtain the following assertion:

Theorem 3. *Let the conditions of Theorem 2 be satisfied; U_x , $U_t U^{-1}$, $r' r^{-1} U$, v_0 have bounded derivatives of second order, and the initial function w_0 be such that for $w_{0y} \equiv w_0(\xi, \eta)$ the smoothness conditions and the compatibility conditions of Lemma 3 are satisfied. Then the solution u, v of problem (1), (2), whose existence is asserted in Theorem 2, has in D , for $t \leq \tau_3$, continuous and bounded derivatives entering system (1).*

We note that Theorems 1, 2, 3 also hold for symmetric two-dimensional flows. The boundary-layer equations for such flows, as is known, have the form of system (1) with $r(x) \equiv 1$ and with conditions (2) on the boundary of the domain D .

2. By the method of lines, exactly as was done above for axisymmetric flows, one can prove the existence of solutions of the Prandtl system for two-dimensional flows and obtain theorems analogous to those obtained earlier in papers ^(1, 2). Consider system (1) for $r(x) > 0$ for $x \geq 0$, or $r(x) \equiv 1$, with the conditions

$$\begin{aligned} u|_{t=0} &= u_0(x, y), \quad u|_{x=0} = u_1(t, y), \quad u|_{y=0} = 0, \quad v|_{y=0} = v_0(t, x), \\ u &\rightarrow U(t, x) \quad \text{as } y \rightarrow \infty. \end{aligned} \tag{11}$$

We assume that $U(t, x) > 0$ for $x \geq 0$, $u_0 > 0$, $u_1 > 0$ for $y > 0$. The change of independent variables (3) leads, for the function $w = u_y/U$, to equation (4), conditions (5), and the condition

$$w|_{\xi=0} = u_{1y} \equiv w_1(\tau, \eta). \tag{12}$$

Consider system (6) for $m = 1, 2, \dots$, $l = 1, 2, \dots$ and adjoin to conditions (7) the condition $w^{m,0} = w_1(mh, \eta)$. For solutions of such a system, under the corresponding assumptions on w_1 , lemmas analogous to Lemmas 1 and 2 are valid, which leads to the following theorem.

Theorem 4. *Let the assumptions of Lemmas 1 and 2 be satisfied with respect to U, v_0, w_0 . Let $K_8(1 - \eta) \leq w_1 \leq K_9(1 - \eta)$, and suppose there exist bounded continuous derivatives $w_{1\eta}, w_{1\tau}, w_1 w_{1\eta\eta}$, with $|w_{1\tau}| \leq K_{10}(1 - \eta)$, $K_i > 0$ constants. Suppose that the compatibility conditions are satisfied: $w_0 = w_1$ for $\tau = 0, \xi = 0$, $\nu w_1 w_{1\eta} - v_0 w_1 + C = 0$ and $\nu w_1^2 w_{1\eta\eta} - w_{1\tau} + A w_{1\eta} + B w_1 = 0$ for $\eta = 0, \xi = 0$. Then in the domain Ω , if $\tau \leq \tau^1$ or if $\xi \leq \xi^1$, there exists a unique solution w of problem (4), (5), (12), having the same properties as the solution w indicated in Theorem 1; τ^1 and ξ^1 are positive constants determined by the data of problem (1), (11).*

Theorem 5. Suppose $U(x) > 0$, and U_x, U_t, v_0 have bounded first derivatives, $u_0 \rightarrow U$, $u_1 \rightarrow U$ as $y \rightarrow \infty$, $u_{0y} > 0$, $u_{1y} > 0$ for $y \geq 0$; $M_4(U - u_0) \leq u_{0y} \leq M_5(U - u_0)$, $M_6(U - u_1) \leq u_{1y} \leq M_7(U - u_0)$, where M_i are positive constants; for u_0 and u_1 there exist bounded derivatives u_{yy}, u_{yyy} , and also $u_{0x}, u_{0xy}, u_{1t}, u_{1ty}$; for u_0 and u_1 the quantities u_{yy}/u_y , $(u_{yyy}u_y - u_{yy}^2)/u_y^2$ are bounded; $|w_{0\xi}| \leq M_8(U - u_0)$, $|w_{1\tau}| \leq M_9(U - u_1)$. Suppose that the compatibility condition (8) is satisfied; $u_0 = u_1$ for $t = 0, x = 0$; $u_1 = 0$, $u_0 = 0$ for $y = 0$, and, moreover, $v_0 u_{1y} = -p_x + \nu u_{1yy}$, $u_{1t} + v_0 u_{1y} = \nu u_{1yyy}$ for $y = 0, x = 0$. Then in the domain D , for $t_0 \leq \tau^1$ or $x_0 \leq \xi^1$, there exists a unique solution of problem (1), (11), having the properties indicated in Theorem 2.

Consider the system

$$uu_x + vu_y = -p_x + \nu u_{yy}, \quad u_x + v_y = 0 \quad (13)$$

in the domain $D_1 \{0 \leq x \leq x_1, 0 \leq y < \infty\}$ with the conditions

$$u|_{x=0} = u_1(y), \quad u|_{y=0} = 0, \quad v|_{y=0} = v_0(x), \quad u \rightarrow U(x) \quad \text{as } y \rightarrow \infty. \quad (14)$$

We shall assume that $U(x) > 0$ for $x \geq 0$, $u_1 > 0$ for $y > 0$, $u_{1y} > 0$ for $y \geq 0$. The change of independent variables (3) and the introduction of the function $w = u_y/U$ reduce problem (13), (14) to the equation

$$\nu w^2 w_{\eta\eta} - \eta U w_{\xi} + (\eta^2 - 1) U_x w w_{\eta} - \eta U_{xw} = 0 \quad (15)$$

in the domain $\Omega_1 \{0 \leq \xi \leq x_1, 0 \leq \eta \leq 1\}$ with the conditions

$$w|_{\xi=0} = u_{1y} \equiv w_1(\eta), \quad w|_{\eta=1} = 0, \quad (\nu w w_{\eta} - v_0 w + U_x)|_{\eta=0} = 0. \quad (16)$$

Let $f^l(\eta) \equiv f(lh, \eta)$. The system of ordinary differential equations

$$\nu(w^{l-1} + h)^2 w_{\eta\eta}^l - (\eta + h)U(w^l - w^{l-1})/h + (\eta^2 - 1)U_x w_\eta^l - \eta U_x w^l = 0, \quad (17)$$

$l = 1, 2, \dots$, $w^0(\eta) = w_1(\eta)$, with the conditions

$$w^l(1) = 0, \quad \nu w^{l-1} w_\eta^l - v_0 w^{l-1} + U_x = 0 \quad \text{for } \eta = 0$$

has the same properties as the system constructed for problem (4), (12), (5). As $h \rightarrow 0$, $w^l(\eta)$ converge to the solution of problem (15), (16), from the solvability of which we obtain a theorem on the existence of a solution of problem (13), (14).

Theorem 6. Suppose U_x, v_0 have bounded first derivatives, and $u_1(y)$ satisfies the smoothness conditions and the compatibility conditions indicated in Theorem 5. Then in the domain D_1 , for $x \leq x^1$, there exists a unique solution u, v of problem (13), (14), having the properties indicated in Theorem 2. Moreover, all derivatives entering into system (13) are continuous at interior points of the domain; x^1 depends on the data of the problem.

For solutions of problems (13), (14) and (1), (11), theorems analogous to Theorem 3 are also valid.

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