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Abstract

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MATHEMATICS

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SOLVABILITY CONDITIONS FOR A CLASS OF PSEUDODIFFERENTIAL EQUATIONS

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1. In the study of the differential operator $P(x, D)$, undoubtedly one of the most important questions is whether the equation

$$P(x, D)u = f \quad (1)$$

is solvable, at least locally and under the strongest assumptions concerning the local regularity of the function f . This question became especially urgent after H. Lewy showed that the answer may also be negative, and L. Hörmander ⁽³⁾ formulated a condition necessary for local solvability of equation (1) in the space of distributions.

On the other hand, there are various sufficient conditions ensuring the existence of solutions of equation (1) in the small. First of all, the existence of solutions was proved for equations with constant coefficients by B. Malgrange and L. Ehrenpreis. Moreover, it is known that equation (1) can be solved locally if the operator $P(x, D)$ is elliptic.

Furthermore, L. Hörmander considered a class of operators $P(x, D)$ whose characteristics at each point $x \in \Omega \subset R^n$ are simple. The latter means that if the operator $P(x, D)$ has order m in Ω and $p(x, D)$ is the homogeneous part of $P(x, D)$ of order m , then

$$\sum_{j=1}^n |\partial p(x, \xi) / \partial \xi_j|^2 \neq 0, \quad 0 \neq \xi \in R^n, \quad x \in \Omega. \quad (2)$$

For such operators L. Hörmander gave a sufficient condition that, for any right-hand side $f \in H^l(\Omega)$, equation (1) have a solution (in the weak sense) $u \in H^{l+m-1}(\Omega)$, provided only that the diameter of the domain Ω is sufficiently small.

Finally, it is known that if the operator $P(x, D)$ with sufficiently smooth coefficients in the domain Ω has constant strength, then in a sufficiently small subdomain $\omega \subset \Omega$ equation (1) has a square-integrable solution for any right-hand side $f \in L_2(\omega)$.

Recently a number of works have appeared devoted to the study of the algebra of pseudodifferential operators ^(1,2). In ⁽²⁾, in particular, some conditions are given that ensure local solvability of equation (1), in which $P(x, D)$ is a pseudodifferential operator.

In the present work we single out a class of pseudodifferential operators $P(x, D)$ which, on the one hand, includes all differential operators of constant strength, and, on the other hand, contains also such operators of nonconstant strength which may have characteristics of arbitrary multiplicity. For operators of this class we formulate an algebraic condition necessary and sufficient for the inequality

$$\|P(x, D)'u\|_s \geq c_s \|u\|_{s+m-1}, \quad u \in C_0^\infty(\Omega_s), \quad (3)$$

where m is the order of the operator $P(x, D)$; $c_s > 0$ is a constant, and Ω_s is a sufficiently small domain in R^n . This condition clearly depends not only on the highest homogeneous part $p(x, D)$, but also on the homogeneous part of order $m - 1$ of the operator $P(x, D)$. It is shown below that, under certain restrictions, the operator $P(x, D)$ satisfies this condition only together with the adjoint operator ${}^tP(x, D)$; hence, from (3) there immediately follows a theorem on the local solvability in $H^{s+m-1}(\Omega_s)$ of equation (1) for any right-hand side $f \in H^s(\Omega_s)$.

2. Let us agree on notation and terminology. Throughout the article we shall denote by R^n the n -dimensional real Euclidean space. Its points will be tuples of n numbers $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$, etc., and $\xi = (\xi_1, \dots, \xi_n)$. Ω will always be a bounded domain in the space R^n of the points x . Further, $D_k = -i\partial/\partial x_k$, and if $\alpha = (\alpha_1, \dots, \alpha_n)$ is a tuple of nonnegative integers, then $D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n}$, and $\xi^\alpha = \xi_1^{\alpha_1} \dots \xi_n^{\alpha_n}$. If $P_k(x, \xi) \in C^\infty(\Omega \times \{R^n \setminus 0\})$ is a function homogeneous of degree k in the vector ξ , then by $P_k(x, D)$ we denote the pseudodifferential operator

$$C_0^\infty(\Omega) \ni u(x) \rightarrow (2\pi)^{-n/2} \int \hat{u}(\xi) P_k(x, \xi) e^{i\langle x, \xi \rangle} d\xi,$$

where $\hat{u}(\xi)$ is the Fourier transform of $u(x)$. In the paper we consider operators P of the form

$$P = P_m(x, D) + P_{m-1}(x, D) + T_k, \quad (4)$$

where T_k is an arbitrary closed linear operator on $C_0^\infty(\Omega)$ of order $k < m - 1$, subject to the following condition.

Definition. There exists a point $x_0 \in \Omega$ such that for any vector a the inequality

$$|D^\alpha P_m(x, \xi)| < c_\alpha |P_m(x_0, \xi)|, \quad x \in \Omega, \quad |\xi| = 1, \quad (5)$$

holds, where c_α is a constant.

It can be shown that the class of operators (4)–(5) is essentially broader than the class of differential operators of constant strength in Ω . In the case where $P_m(x, \xi)$ is a polynomial in ξ , condition (5) for $\alpha = 0$ means that the algebraic cone of normals of the operator P does not depend on the point x (in a neighborhood of x_0).

Theorem 1. Let P be an operator of the form (4)–(5). If

$$\begin{aligned} & |P_{m-1}(x_0, \xi)| + \sum_1^n \left| \frac{\partial P_m(x_0, \xi)}{\partial \xi_j} \right| \neq 0, \quad \text{when } P_m(x_0, \xi) = 0, \\ & \max \left\{ \frac{\operatorname{Re} P_m(x_0, \xi) \overline{P}_{m-1}(x_0, \xi)}{|P_m(x_0, \xi)|}, \quad \sum_1^n \left| \frac{\partial P_m(x_0, \xi)}{\partial \xi_j} \right| + \frac{|\operatorname{Im} P_m(x_0, \xi) \overline{P}_{m-1}(x_0, \xi)|}{|P_m(x_0, \xi)|} \right\} \geq \\ & \geq c > 0, \quad \text{when } P_m(x_0, \xi) \neq 0, \end{aligned} \quad (6)$$

then there exists a neighborhood ω of the point x_0 such that for all functions $u \in C_0^\infty(\omega)$ the inequality

$$\|Pu\|_s \geq \operatorname{const} \|u\|_{s+m-1} \quad (7)$$

holds.

Conversely, from inequality (7) there follows the validity of conditions (6) for all $x \in \omega$.

For the proof of local solvability of equation (1), where the operator P satisfies conditions (4)–(5), we need to use information about the operator adjoint to P , which is defined as the operator ${}^tP$ such that

$$\int (Pu)v \, dx = \int u({}^tPv) \, dx, \quad u, v \in C_0^\infty(\Omega).$$

Theorem 2. If an operator P of the form (4)–(5) satisfies the condition

$$P_{m-1}(x_0, \xi) - i \sum_1^n \frac{\partial}{\partial \xi_j} \frac{\partial}{\partial x_j} P_m(x_0, \xi) \neq 0, \quad \text{when } \sum_{j=1}^n \left| \frac{\partial P_m(x_0, \xi)}{\partial \xi_j} \right| = 0,$$

then the operator ${}^tP$ satisfies condition (6), and, consequently, estimate (3) holds for it.

Thus, the operators $P(x, D)$ and ${}^tP(x, D)$ can satisfy the restrictions imposed below only simultaneously, and we easily prove the final theorem.

Theorem 3. Let P be an operator for which the condition of Theorem 2 is satisfied at some point $x_0 \in \Omega$. Then there exists a neighborhood ω_s of this point in which the equation $P(x, D)u = f$ is solvable in $H^{s+m-1}(\omega_s)$ for any function $f \in H^s(\omega_s)$.

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Note: Figure translations are in progress. See original paper for figures.

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