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Abstract

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MATHEMATICS

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ON THE EQUATION $\Delta U[x(t)] + P[x(t)]U[x(t)] = 0$ IN FUNCTION SPACE

(Presented by Academician A. Yu. Ishlinskii on 16 IV 1966)

The Laplacian of a functional $\Delta U[x(t)]$ —the continual analogue of the Laplace operator of a function of a finite number of variables—was defined by P. Lévy⁽¹⁾. Laplace and Poisson equations with such Laplacians were studied by P. Lévy, E. M. Polishchuk⁽²⁾, and the author^(3,4). In this note we consider the boundary-value problem of the “generalized Laplace equation”

$$\Delta U[x(t)] + P[x(t)]U[x(t)] = 0 \quad (1)$$

for a domain $\Omega \cup \Gamma$ in the space C , defined by the inequality $S[x(t)] \leq 1$ ($x(t) \in \Omega$, if $S[x(t)] < 1$; $x(t) \in \Gamma$, if $S[x(t)] = 1$); $S[x(t)]$ is a twice functionally differentiable functional such that its first and second variations have normal form, and $\Delta S[x(t)]$ is a constant positive nonzero number; C is the space of functions $x(t)$ continuous on $[0, 1]$ ($x(0) = 0$), with Wiener measure⁽⁵⁾.

1. Let $P[x(t)]$ be a functional of finite degree⁽⁶⁾; $P_N[x(t)]$ the partial sum of its expansion in the Fourier-Hermite functionals⁽⁷⁾; $B_{m_1 \dots m_N}$ its Fourier-Hermite coefficients;

$$\Psi_{m_1 \dots m_N}[x(t)] = \prod_{i=1}^N H_{m_i} \left[\int_0^1 \chi_i(t) dx(t) \right]$$

are the Fourier-Hermite functionals ($m_i = 0, 1, 2, \dots$; $N = 1, 2, \dots$); $H_m(u)$ are partially normalized Hermite polynomials ($m = 0, 1, 2, \dots$); $\chi_1(t) \equiv \chi_0^{(0)}(t)$, $\chi_i(t) \equiv \chi_n^{(k)}(t)$ for $i = 2^n + k$ ($n = 0, 1, 2, \dots$; $k = 1, 2, \dots, 2^n$), $\chi_0^{(0)}(t)$, $\{\chi_n^{(k)}(t)\}$ are the Haar system of functions. Let

$$\begin{aligned} \Psi_{m_1 \dots m_N}^*[x(t), y_1(\tau), \dots, y_N(\tau)] = \\ = \prod_{i=1}^N H_{m_i} \left[\int_0^1 \chi_i(t) dx(t) + 2(1 - S[x])^{1/2} (\Delta S[x])^{-1/2} \tilde{y}_i(\tau) \right], \end{aligned}$$

$$\tilde{y}_1(\tau) \equiv y_1(\tau), \quad \tilde{y}_i(\tau) \equiv \tilde{y}_n^{(k)}(\tau), \quad \tilde{y}_n^{(k)}(\tau) = \sqrt{2^n} \left(2z_{(2k-1)/2^{n+1}}(\tau) - z_{(2k-2)/2^{n+1}}(\tau) - z_{2k/2^{n+1}}(\tau) \right),$$

$$z_0(\tau) = 0, \quad z_{p_q}(\tau) = y_q(\tau), \quad (p_q = 1, 1/2, 1/4, 3/4, \dots; q = 1, 2, \dots, N).$$

Lemma 1. The functional

$$\begin{aligned} \Phi_{m_1 \dots m_N, N}[x(t)] &= \int_{C_N} \Psi_{m_1 \dots m_N}^*[x(t), y_1(1), \dots, y_N(1)] \times \\ &\times \exp \left\{ \frac{1 - S[x]}{\Delta S[x]} \int_0^1 \sum_{\mu_1, \dots, \mu_N=0}^N B_{\mu_1 \dots \mu_N} \Psi_{\mu_1 \dots \mu_N}^*[x(t), y_1(s), \dots, y_N(s)] ds \right\} dw_{y_1} \dots dw_{y_N}, \end{aligned}$$

where the integral is understood as an N -fold Wiener integral on the product space C , satisfies in Ω the equation

$$\Delta U[x] + P_N[x]U[x] = 0.$$

Indeed, having found the second variation and then computed the Laplacian of the functional $\Phi_{m_1 \dots m_N, N}[x]$, we obtain

$$\begin{aligned}
\Delta\Phi_{(m),N}[x] = \int_{C^N} & \left\{ \sum_{i=1}^N \left[2^{n_i+1} \sqrt{m_i(m_i-1)} a_i \Psi_{(m)-2_i}^*(1) \right. \right. \\
& - (1-S)^{-1/2} (\Delta S)^{1/2} \sqrt{2m_i} \tilde{y}_i(1) \Psi_{(m)-1_i}^*(1) \\
& \left. \left. + \sum_{\substack{l=1 \\ l \neq i}}^N \sqrt{2^{n_i+n_l+2} m_i m_l} a_{il} \Psi_{(m)-1_i-1_l}^*(1) \right] \right. \\
& + 2(1-S)(\Delta S)^{-1} \sqrt{2^{n_i+1} m_i} b_i \Psi_{(m)-1_i}^*(1) \\
& \times \left[\int_0^1 \sum_{(\mu)=0}^N B_{(\mu)} \sum_{l=1}^N \sqrt{2^{n_l+1} \mu_l} b_l \Psi_{(\mu)-1_l}^*(s) ds \right] - \Psi_{(m)}^*(1) \int_0^1 \sum_{(\mu)=0}^N B_{(\mu)} \Psi_{(\mu)}^*(s) ds \\
& + (1-S)(\Delta S)^{-1} \Psi_{(m)}^*(1) \int_0^1 \sum_{(\mu)=0}^N B_{(\mu)} \sum_{i=1}^N \left[2^{n_i+1} \sqrt{\mu_i(\mu_i-1)} a_i \Psi_{(\mu)-2_i}^*(s) \right. \\
& - (1-S)^{-1/2} (\Delta S)^{1/2} \sqrt{2\mu_i} \tilde{y}_i(s) \Psi_{(\mu)-1_i}^*(s) \\
& \left. \left. + \sum_{\substack{l=1 \\ l \neq i}}^N \sqrt{2^{n_i+n_l+2} \mu_i \mu_l} a_{il} \Psi_{(\mu)-1_i-1_l}^*(s) \right] ds \right. \\
& \left. + (1-S)^2 (\Delta S)^{-2} \Psi_{(m)}^*(1) \left[\int_0^1 \sum_{(\mu)=0}^N B_{(\mu)} \sum_{i=1}^N \sqrt{2^{n_i+1} \mu_i} B_i \Psi_{(\mu)-1_i}^*(s) ds \right]^2 \right\} E_N dw y_1 \dots du
\end{aligned} \tag{2}$$

where

$$\Psi_{m_1 \dots m_N}^*(\tau) = \Psi_{m_1 \dots m_N}^*[x(t), y_1(\tau), \dots, y_N(\tau)], \quad (m) = (m_1, \dots, m_N),$$

$$(m) - q_i = (m_1, \dots, m_{i-1}, m_i - q, m_{i+1}, \dots, m_N),$$

and a_i, a_{il}, b_i are the sums of the coefficients at the corresponding $[\delta x]^2$,

$$E_N = \exp \left\{ \frac{1-S}{\Delta S} \int_0^1 \sum_{(\mu)=0}^N B_{(\mu)} \Psi_{(\mu)}^*(s) ds \right\}.$$

Computing, by P. Cameron's theorem (8), the Wiener integral of the first partial variation with respect to y_i of the functional

$$(i-S)^{-1/2} (\Delta S)^{1/2} \sqrt{2m_i} \Psi_{(m)-1_i}^*(1) E_N,$$

we have

$$\begin{aligned}
& (i-S)^{-1/2}(\Delta S)^{1/2} \int_{C^N} \sum_{i=1}^N \sqrt{2m_i} \tilde{y}_i(1) \Psi_{(m)-1_i}^*(1) E_N dw_{y_1} \dots dw_{y_N} \\
&= \int_{C^N} \sum_{i=1}^N \left\{ 2^{n_i+1} \sqrt{m_i(m_i-1)} a_i \Psi_{(m)-2_i}^*(1) + \sum_{\substack{l=1 \\ l \neq i}}^N \sqrt{2^{n_i+n_l+2} m_i m_l} a_{il} \Psi_{(m)-1_i-1_l}^*(1) \right. \\
&\quad \left. + (1-S)(\Delta S)^{-1} \sqrt{2^{n_i+1} m_i} b_i \Psi_{(m)-1_i}^*(1) \int_0^1 s \sum_{(\mu)=0}^N B_{(\mu)} \sum_{l=1}^N \sqrt{2^{n_l+1} \mu_l} b_l \Psi_{(\mu)-1_l}^*(s) ds \right\} \\
&\quad \times E_N dw_{y_1} \dots dw_{y_N},
\end{aligned} \tag{3}$$

and by M. Ouchard' s theorem (9, 10)—for the partial variational derivative of the functional

$$(1-S)^{1/2}(\Delta S)^{-1/2} \Psi_{(m)}^*(1) \sqrt{2\mu_i} \Psi_{(\mu)-1_i}^*(s) E_N$$

with respect to y_i —we obtain

$$\begin{aligned}
& (1-S)^{1/2}(\Delta S)^{-1/2} \int_{C^N} \Psi_{(m)}^*(1) \left[\int_0^1 \sum_{(\mu)=0}^N B_{(\mu)} \sum_{i=1}^N \sqrt{2\mu_i} \tilde{y}_i(s) \Psi_{(\mu)-1_i}^*(s) ds \right] E_N dw_{y_1} \dots dw_{y_N} \\
&= \int_{C^N} \left\{ -2(1-S)^2(\Delta S)^{-2} \Psi_{(m)}^*(1) \int_0^s \sum_{(\mu)=0}^N B_{(\mu)} \sum_{i=1}^N \sqrt{2^{n_i+1} \mu_i} b_i \Psi_{(\mu)-1_i}^*(s) \right. \\
&\quad \times \int_1^s \sum_{(\mu)=0}^N B_{(\mu)} \sum_{l=1}^N \sqrt{2^{n_l+1} \mu_l} b_l \Psi_{(\mu)-1_l}^*(v) dv ds + \\
&\quad + (1-S)(\Delta S)^{-1} \Psi_{(m)}^*(1) \left\{ \int_0^1 s \sum_{(\mu)=0}^N B_{(\mu)} \sum_{i=1}^N \left[2^{n_i+1} \sqrt{\mu_i(\mu_i-1)} a_i \Psi_{(\mu)-2_i}^*(s) \right. \right. \\
&\quad \left. \left. + \sum_{\substack{l=1 \\ l \neq i}}^N \sqrt{2^{n_i+n_l+2} \mu_i \mu_l} a_{il} \Psi_{(\mu)-1_i-1_l}^*(s) \right] ds \right\} \\
&\quad + (1-S)(\Delta S)^{-1} \sum_{i=1}^N \sqrt{2^{n_i+1} m_i} b_i \Psi_{(m)-1_i}^*(1) \times \\
&\quad \times \left\{ \int_0^1 s \sum_{(\mu)=0}^N B_{(\mu)} \sum_{l=1}^N \sqrt{2^{n_l+1} \mu_l} b_l \Psi_{(\mu)-1_l}^*(s) ds \right\} E_N^2 dw_{y_1} \dots dw_{y_N}.
\end{aligned} \tag{4}$$

Extending lemma (10) to the N -dimensional case and using E. Ouchar's theorem, it is not difficult to obtain the relation

$$\begin{aligned}
& \int_{C^N} \Psi_{(m)}^*(1) \int_0^1 \sum_{(\mu)=0}^N B_{(\mu)} \Psi_{(\mu)}^*(s) ds E_N dw y_1 \dots dw y_N = P_N[x] \Phi_{(m),N}[x] - \\
& - \int_{C^N} \left\{ 2(1-S)^2 (\Delta S)^{-2} \Psi_{(m)}^*(1) \int_0^1 (1-s) \sum_{(\mu)=0}^N B_{(\mu)} \sum_{i=1}^N \sqrt{2^{n_i+1}} \mu_i b_i \Psi_{(\mu)-1_i}^*(s) \times \right. \\
& \times \int_1^s \sum_{(\mu)=0}^N B_{(\mu)} \sum_{l=1}^N \sqrt{2^{n_l+1}} \mu_l b_l \Psi_{(\mu)-1_l}^*(v) dv ds - \\
& - (1-S) (\Delta S)^{-1} \Psi_{(m)}^*(1) \int_0^1 (1-s) \sum_{(\mu)=0}^N B_{(\mu)} \sum_{i=1}^N \left[2^{n_i+1} \sqrt{\mu_i(\mu_i-1)} a_i \Psi_{(\mu)-2_i}^*(s) + \right. \\
& \left. + \sum_{\substack{l=1 \\ l \neq i}}^N \sqrt{2^{n_i+n_l+2}} \mu_i \mu_l a_{il} \Psi_{(\mu)-1_i-1_l}^*(s) \right] ds - \\
& - 2(1-S) (\Delta S)^{-1} \sum_{i=1}^N \sqrt{2^{n_i+1}} m_i b_i \Psi_{(m)-1_i}^*(1) \times \\
& \left. \times \int_0^1 (1-s) \sum_{(\mu)=0}^N B_{(\mu)} \sum_{l=1}^N \sqrt{2^{n_l+1}} \mu_l b_l \Psi_{(\mu)-1_l}^*(s) ds \right\} E_N dw y_1 \dots dw y_N.
\end{aligned} \tag{5}$$

Substituting now (3), (4), and (5) into (2), we obtain that $\Phi_{(m),N}[x] = -P_N(x) \Phi_{(m),N}[x]$ in the domain Ω .

From Lemma 2 and Corollary 1 ⁽³⁾ it follows that

Lemma 2. If $P[x(t)] < 0$, $\Omega_h \cup \Gamma_h$ is the closure of the set of functions

$$x_h(t) = \frac{1}{2h} \int_{t-h}^{t+h} x(s) ds, \quad x(t) \in \Omega \cup \Gamma, \quad x(t) = 0 \quad \text{outside } [0, 1],$$

then a functional $V[x(t)]$, satisfying equation (1) in Ω_h and continuous in $\Omega_h \cup \Gamma_h$ for any $h > 0$, cannot attain a positive maximum and a negative minimum in Ω_h .

Corollary. If $P[x(t)] \leq 0$, and the functionals $V_1[x(t)]$ and $V_2[x(t)]$ satisfy equation (1) in Ω and Ω_h , are continuous in $\Omega \cup \Gamma$ and $\Omega_h \cup \Gamma_h$, and $V_1|_{\Gamma} = V_2|_{\Gamma} = H[x]$, then $V_1 \equiv V_2$ in Ω .

2. Theorem 1. Suppose a finite-degree functional $G[x(t)]$ is given; $A_{m_1 \dots m_N}$ are its Fourier-Hermite coefficients; $P[x(t)]$ is a finite-degree functional, $P[x(t)] \leq 0$.

Then in the domain Ω there exists a unique solution of equation (1), coinciding with the given functional $G[x(t)]$ on the surface Γ , which is equal to

$$U[x(t)] = \lim_{N \rightarrow \infty} \sum_{m_1, \dots, m_N=0}^N A_{m_1 \dots m_N} \Phi_{m_1 \dots m_N, N}[x(t)] \quad (6)$$

for almost all $x(t) \in \Omega \cup \Gamma$.

Let us now consider the solution of equation (1) in the space $L_2(C)$.

Theorem 2. Let a functional $G[x(t)] \in L_2(C)$ be given; let $A_{m_1 \dots m_N}$ be its Fourier-Hermite coefficients; let $P[x(t)]$ be a functional of finite degree, bounded below almost everywhere, $P[x(t)] \leq 0$. Then in the domain Ω there exists a unique solution $U[x(t)] \in L_2(C)$

$$U[x(t)] = \text{L.I.M.}_{N \rightarrow \infty} \sum_{m_1, \dots, m_N=0}^N A_{m_1 \dots m_N} \Phi_{m_1 \dots m_N, N}[x(t)] \quad (7)$$

of equation (1), equal to the prescribed functional $G[x(t)]$ on the surface Γ .

The proof of the theorems follows from the properties of the functionals $G[x(t)]$, $P[x(t)]$, Lemma 1, and the corollary.

We note that, putting $P[x(t)] \equiv 0$ in (6) and (7), we obtain the solution of the boundary-value problem for the "Laplace equation." It coincides with the solution given in the author's paper ⁽³⁾, if one computes the multiple Wiener integral of the functional concentrated at a point and then uses the Paley-Wiener formula ⁽¹¹⁾.

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Note: Figure translations are in progress. See original paper for figures.

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