

# ON THE INDICATOR OF FUNCTIONS OF INTEGER ORDER, ANALYTIC AND OF COMPLETELY REGULAR GROWTH IN A HALF-PLANE

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1967

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**Abstract**

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UDC 517.544

MATHEMATICS

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# ON THE INDICATOR OF FUNCTIONS OF INTEGER ORDER, ANALYTIC AND OF COMPLETELY REGULAR GROWTH IN A HALF-PLANE

*(Presented by Academician S. N. Bernstein on 8 IV 1966)*

B. Ya. Levin and A. Pfluger showed ((<sup>1</sup>), Chs. I, III; (<sup>2,3</sup>)) that an entire function of order  $\rho > 0$  has completely regular growth if and only if the set of its zeros  $\{z_n\}$  is regularly distributed, i.e.:

1. For all  $\theta_k \in [0, 2\pi] \setminus N$ , where  $N$  is at most countable, there exists the limit

$$\lim_{r \rightarrow \infty} \frac{n(r, \theta_1, \theta_2)}{r^\rho} = \Delta(\theta_1, \theta_2), \quad (1)$$

called the angular density (see (<sup>1</sup>), p. 118).

2. In the case of integer  $\rho$ , the zeros  $z_n$  are arranged with a special symmetry, namely, there exists the limit

$$\lim_{r \rightarrow \infty} \sum_{|z_n| < r} z_n^{-\rho} = \sigma \neq \infty. \quad (2)$$

In the works of the authors cited, a formula for the indicator of these entire functions was also obtained. In article (<sup>4</sup>) these results were extended to functions of noninteger order, regular in a half-plane. For them, the concept of argument density, which is an analogue of angular density, is introduced in a special way, and a formula for the indicator is obtained. In the present note we consider functions of integer order in a half-plane. Definitions of the order of a function and of functions of completely regular growth (in an open or closed angle) are given in article (<sup>4</sup>).

**Theorem 1.** *Every function, regular and of integer order  $\rho \geq 0$  in  $\text{Im } z > 0$ , is uniquely representable in the form*

$$\begin{aligned}
 f(z) = \exp \left\{ i \sum_{k=0}^{\rho} a_k z^k + \frac{1}{\pi i} \left[ \int_{-1}^1 \frac{\ln |f(t)|}{t-z} dt + \int_{-1}^1 \frac{i d\varphi(t)}{t-z} \right. \right. \\
 \left. \left. + z^{\rho+1} \int_{|t|>1} \frac{\ln |f(t)| dt}{t^{\rho+1}(t-z)} + z^{\rho+1} \int_{|t|>1} \frac{d\varphi(t)}{t^{\rho+1}(t-z)} \right] \right\} \prod_{|z_n| \leq 1} \frac{z - z_n}{z - \bar{z}_n} \times \\
 \times \prod_{|z_n| > 1} \frac{\left(1 - \frac{z}{z_n}\right) \exp \left[ \frac{z}{z_n} + \frac{1}{2} \left(\frac{z}{z_n}\right)^2 + \dots + \frac{1}{\rho} \left(\frac{z}{z_n}\right)^{\rho} \right]}{\left(1 - \frac{z}{\bar{z}_n}\right) \exp \left[ \frac{z}{\bar{z}_n} + \frac{1}{2} \left(\frac{z}{\bar{z}_n}\right)^2 + \dots + \frac{1}{\rho} \left(\frac{z}{\bar{z}_n}\right)^{\rho} \right]}, \quad (3)
 \end{aligned}$$

where  $a_k$  are real constants;  $z_n$  are the interior (lying in  $\text{Im } z > 0$ ) zeros of  $f(z)$ ;  $\varphi(t)$  is a real nondecreasing function for which  $\varphi'(t) = 0$  almost everywhere and the integral

$$\int_{|t| \geq 1} \frac{d\varphi(t)}{t^{\rho+2}}$$

converges.

The theorem is readily obtained from the analogous theorem given in (4).\*

\* For the case of integer order, Theorem 1 is in a certain sense more convenient than the one mentioned ( $\varphi(t)$  and  $a_k$  in these theorems are different).

Starting from representation (3), for an arbitrary function  $f(z)$ , regular and of integer order  $\rho > 0$  in  $\text{Im } z > 0$ , we introduce the following notation:

$$\tau(t) = \begin{cases} \frac{1}{2\pi} \int_1^t \frac{\ln |f(x)|}{x} dx + \frac{1}{2\pi} \int_1^t \frac{d\varphi(x)}{x}, & t > 1, \\ \frac{1}{2\pi} \int_t^{-1} \frac{\ln |f(x)|}{|x|} dx + \frac{1}{2\pi} \int_t^{-1} \frac{d\varphi(x)}{|x|}, & t < -1, \end{cases} \quad (4)$$

$$c(r, \eta_1, \eta_2) = \sum_{\eta_1 < \theta_n \leq \eta_2, 1 \leq r_n \leq r} \sin \theta_n, \quad r > 1 \quad (z_n = r_n e^{i\theta_n}), \quad (5)$$

$$a(r, \eta_1, \eta_2) = \begin{cases} c(r, \eta_1, \eta_2), & 0 < \eta_1 < \eta_2 < \pi, \\ c(r, 0, \eta_2) - \tau(r), & 0 = \eta_1 < \eta_2 < \pi, \\ c(r, \eta_1, \pi) - \tau(-r), & 0 < \eta_1 < \eta_2 = \pi, \\ c(r, 0, \pi) - \tau(r) - \tau(-r), & \eta_1 = 0, \eta_2 = \pi, \end{cases} \quad (6)$$

$$a(r, \eta_1, \eta_2) = -a(r, \eta_2, \eta_1), \quad \eta_1 > \eta_2; \quad a(r, \eta, \eta) \equiv 0.$$

**Definition 1.** If for a function  $f(z)$ , regular and of order  $\rho > 0$  in  $\text{Im } z > 0$ , for all  $\eta_1, \eta_2 \in [0 \leq \theta \leq \pi] \setminus N$ , where  $N$  is at most countable and contains no points  $\eta = 0, \eta = \pi$ , there exists the finite limit

$$\lim_{r \rightarrow \infty} \frac{a(r, \eta_1, \eta_2)}{r^\rho} = \lambda(\eta_1, \eta_2),$$

then we shall say that the set of zeros of the function  $f(z)$  has an **argument-boundary density** in the half-plane  $\text{Im } z > 0$ .

**Definition 2.** A function  $f(z)$  of order  $\rho > 0$  in  $\text{Im } z > 0$  is called a **function of finite type** if, asymptotically,

$$\sup_{|z| \leq r, \text{Im } z > 0} |f(z)| < \exp(Kr^\rho), \quad K = \text{const.} \quad (7)$$

Denote by  $A_\rho$  (by  $\overline{A}_\rho$ ) the class of functions regular of order  $\rho$  and of finite type in  $\text{Im } z > 0$ , having completely regular growth in the open angle  $0 < \arg z < \pi$  (in the closed angle  $0 \leq \arg z \leq \pi$ ).

**Theorem 2.** Let the function  $f(z)$  be regular of integer order  $\rho > 0$  and of finite type in  $\text{Im } z > 0$ . Then, in order that  $f(z)$  belong to the class  $A_\rho$ , it is necessary and sufficient that the set of its zeros have an argument-boundary density and that there exist the finite limit

$$\sigma = \lim_{r \rightarrow \infty} \left[ \frac{1}{\rho} \sum_{1 \leq r_n \leq r} \frac{\sin \rho \theta_n}{r_n^\rho} - \frac{1}{2\pi} \int_{1 \leq |x| \leq r} \frac{\ln |f(x)|}{x^{\rho+1}} dx - \frac{1}{2\pi} \int_{1 \leq |x| \leq r} \frac{d\varphi(x)}{x^{\rho+1}} \right]. \quad (8)$$

**Theorem 3.** If  $f(z) \in A_\rho$  and  $\rho > 0$  is an integer, then the indicator of the function  $f(z)$  is expressed by the formula

$$\begin{aligned} h_f(\theta) = & (2\sigma - a_\rho) \sin \rho \theta + 2 \cos \rho \theta \int_0^\pi \psi \frac{\sin \rho \psi}{\sin \psi} d\lambda(\psi) + \\ & + 2 \int_0^\theta \frac{\psi}{\sin \psi} \sin \rho(\theta - \psi) d\lambda(\psi) + 2 \int_\theta^\pi \frac{\psi - \pi}{\sin \psi} \sin \rho(\theta - \psi) d\lambda(\psi), \end{aligned} \quad (9)$$

where  $\lambda(\psi) \equiv \lambda(0, \psi)$  is the argument-boundary density of the zeros of  $f(z)$ , and  $\sigma$  and  $a_\rho$  are defined respectively by relations (8) and (3).

**Remark.** For writing formula (9) in other forms, Theorem 4 may be useful.

**Theorem 4.** If  $\lambda(\psi) = \lambda(0, \psi)$  is the argument-boundary density of the function  $f(z) \in A_\rho$ ,  $\rho > 0$  an integer, then the equality

$$\int_0^\pi \frac{\sin \rho \psi}{\sin \psi} d\lambda(\psi) = 0 \quad (10)$$

holds.

**Definition 3.** Let the function  $f(z)$  be regular and of order  $\rho > 0$  in  $\text{Im } z > 0$ , and let  $\tau(r)$  be defined by equality (4). Then the limits

$$\lim_{r \rightarrow +\infty} \frac{\tau(r)}{r^\rho} = l_1, \quad \lim_{r \rightarrow +\infty} \frac{\tau(-r)}{r^\rho} = l_2, \quad (11)$$

if they exist and are finite, are called, respectively, the right- and left-hand boundary density of the set of zeros of the function  $f(z)$ . If in (11) upper limits are taken, then we shall call them upper boundary densities ( $l_1^*$  and  $l_2^*$ ).

**Definition 4.** Let a set of points  $\{z_n\}$ ,  $n = 1, 2, \dots$ , be given in  $\text{Im } z > 0$ , all of whose limit points lie on the real axis. Then, if for all  $\eta_1, \eta_2 \in (0 \leq \eta \leq \pi) \setminus N$ , where  $N$  is at most countable and does not contain  $\eta = 0, \eta = \pi$ , there exists the finite limit

$$\lim_{r \rightarrow \infty} \frac{c(r, \eta_1, \eta_2)}{r^\rho} = \mu(\eta_1, \eta_2), \quad (12)$$

then we shall say that the set  $\{z_n\}$  has, in the domain  $\text{Im } z > 0$ , an argument density with exponent  $\rho$  (or an upper argument density  $\mu^*(\eta_1, \eta_2)$ , if in (12) an upper limit is taken).

**Theorem 5.** If  $f(z)$  is a function of order  $\rho \geq 1$  and of finite type in  $\text{Im } z > 0$ , then its upper boundary densities are finite, and its upper argument density is bounded.

**Theorem 6.** If  $f(z)$  is a function of class  $A_\rho$  and  $\rho > 0$  is an integer, then the indicator of  $f(z)$  can be expressed by the formula ( $0 < \theta < \pi$ )

$$h_f(\theta) = 2 \left\{ \sin \rho \theta \cdot \left[ \sigma - l_1^* + (-1)^\rho l_2^* - \frac{a_\rho}{2} \right] + \cos \rho \theta \cdot \left[ \int_0^\pi \psi \frac{\sin \rho \psi}{\sin \psi} d\mu^*(\psi) + \pi \rho (-1)^\rho l_2^* \right] \right. \\ \left. + \int_0^\theta \frac{\psi}{\sin \psi} \sin \rho(\theta - \psi) d\mu^*(\psi) + \int_\theta^\pi \frac{\psi - \pi}{\sin \psi} \sin \rho(\theta - \psi) d\mu^*(\psi) \right\}.$$

**Theorem 7.** Let  $f(z)$  be regular in  $\text{Im } z > 0$  and continuous in  $\text{Im } z \geq 0$ , and have integer order  $\rho > 0$  and finite type. Then  $f(z) \in A_\rho$  if and only if:

1. The zeros of  $f(z)$  have boundary and argument density, the latter being continuous at  $\psi = 0$  and  $\psi = \pi$ .
2. The function  $\varphi(t)$ , defined from (3), satisfies the equality

$$\lim_{r \rightarrow \infty} \frac{1}{r^\rho} \int_{1 \leq |t| \leq r} \frac{d\varphi(t)}{|t|} = 0. \quad (13)$$

3. There exists the finite limit (8).
4. The indicator  $h_f(\theta)$  is continuous at  $\theta = 0$  and  $\theta = \pi$ .

The indicator of the function  $f(z) \in \overline{A}_\rho$  has the following form ( $0 \leq \theta \leq \pi$ ):

$$h_f(\theta) = 2 \left\{ \sin \rho \theta \cdot \left[ \sigma - l_1 + (-1)^\rho l_2 - \frac{a_\rho}{2} \right] + \cos \rho \theta \cdot \left[ \int_0^\pi \psi \frac{\sin \rho \psi}{\sin \psi} d\mu(\psi) + \pi \rho (-1)^\rho l_2 \right] \right. \\ \left. + \int_0^\theta \frac{\psi}{\sin \psi} \sin \rho(\theta - \psi) d\mu(\psi) + \int_\theta^\pi \frac{\psi - \pi}{\sin \psi} \sin \rho(\theta - \psi) d\mu(\psi) \right\}.$$

**Theorem 8.** If the assumptions of Theorem 7 are satisfied and  $\rho$  is odd, then  $f(z) \in \overline{A}_\rho$  if and only if:

1. The set of zeros of the function  $f(z)$  has, in the open angle  $0 < \theta < \pi$ , an angular density (see (1), where it is assumed that  $0 < \theta_k < \pi$ ).
2. The limit (8) exists.
3. The indicator  $h_f(\theta)$  satisfies the condition

$$h_f(0) + h_f(\pi) = 2\pi \int_{+0}^{\pi-0} \sin \rho \psi d\Delta(\psi).$$

We note that the last improper integral certainly converges for every function of the class  $A_\rho$  ((<sup>4</sup>, p. 498)).

In conclusion we give two theorems concerning functions of finite degree, i.e., functions of order not exceeding one and of finite type.

**Definition 5.** A function analytic in  $\text{Im } z \geq 0$ , whose zeros  $z_n = r_n e^{i\theta_n}$  satisfy the condition (see (<sup>1</sup>, p. 289))

$$\sum_{n=1}^{\infty} \frac{\sin \theta_n}{r_n} < \infty,$$

is called a function of the class  $A$ .

**Theorem 9.** In order that a function  $f(z)$ , regular and of finite degree in  $\text{Im } z \geq 0$ , belong simultaneously to the classes  $A$  and  $A_1$ , it is necessary and

sufficient that the limits

$$\lim_{r \rightarrow \infty} \frac{1}{r} \int_1^r \frac{\ln |f(-x)|}{x} dx = l_2, \quad \lim_{r \rightarrow \infty} \frac{1}{r} \int_1^r \frac{\ln |f(x)|}{x} dx = l_1$$

(the left and right boundary densities) exist, and that the integral

$$\int_1^\infty \frac{\ln |f(x)f(-x)|}{x^2} dx \quad (14)$$

converge.

**Theorem 10.** In order that a function, regular and of finite degree in  $\text{Im } z \geq 0$ , belong simultaneously to the classes  $A$  and  $\bar{A}_1$ , it is necessary and sufficient that the integral (14) converge and that the condition

$$h_f(0) + h_f(\pi) = 0.$$

be satisfied.

It must be noted that, for the case of sufficiency, Theorem 10 was established earlier (see (<sup>1</sup>, p. 317)).

In conclusion I express my deep gratitude to Prof. F. D. Gakhov, who supervised the present work.

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Received

1 IV 1966

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*Note: Figure translations are in progress. See original paper for figures.*

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