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Abstract

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MATHEMATICS

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ON A MAXIMUM PRINCIPLE FOR HYPERBOLIC EQUATIONS AND ITS APPLICATION TO EQUATIONS OF MIXED TYPE

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§ 1. Consider the equation

$$\mathcal{L}[u] \equiv u_{\xi\eta} + a(\xi; \eta)u_{\xi} + b(\xi; \eta)u_{\eta} + c(\xi; \eta)u = 0 \quad (\mathcal{L})$$

in the characteristic triangle $O_0A_0C_0$ (the domain Δ , $O_0(0; 0)$, $A_0(\xi_0; -\xi_0)$, $C_0(0; -\xi_0)$, $\xi_0 > 0$), $a, a_{\xi}, b, c \in C^{(0)}(\Delta \setminus \overline{O_0A_0})$. Let

$$h = a_{\xi} + ab - c = h_1 + h_2, \quad h_1 \leq 0 \text{ in } \Delta; \quad \beta = \exp \left\{ \int_0^{\xi} b d\xi \right\}; \quad \alpha = \beta a;$$

$$\gamma = -\beta h_1 - \beta h_2 = \gamma_1 + \gamma_2; \quad c = c_1 + c_2, \quad c_1 \leq 0 \text{ in } \Delta.$$

We shall say that the coefficients of $(\mathcal{L})$ satisfy in Δ conditions (C) if: 1) $a < 0$ on $\overline{O_0C_0} \setminus O_0$; 2)

$$-a(P) > \int_{PQ} [2|\gamma_2| + \beta|c_2|] d\xi$$

in $\Delta \cup C_0A_0$, and conditions (D), if in $\Delta \cup C_0A_0$: 1) $a < 0$, 2)

$$-\beta(Q)a(Q) > \int_{PQ} |\gamma| d\xi,$$

where $P \in \overline{O_0C_0} \setminus O_0$, PQ is a segment of the characteristic $\eta = \text{const}$. We shall call a function $u(\xi; \eta)$ a solution of equation $(\mathcal{L})$ of class $[R]$ if $\mathcal{L}[u] \equiv 0$ in Δ , $u \in C^{(2)}(\Delta)$, $u \in C^{(0)}(\bar{\Delta})$, $u \in C^{(1)}(\bar{\Delta} \setminus \overline{O_0A_0})$, $u|_{\overline{O_0C_0}} = 0$,

$$\max_{\bar{\Delta}} u > 0, \quad \max_{\bar{\Delta}} u \geq -\min_{\bar{\Delta}} u.$$

Theorem 1 (maximum principle). *Let the coefficients of $(\mathcal{L})$ satisfy in Δ conditions (C) or (D), and let $u(\xi; \eta)$ be an arbitrary solution of $(\mathcal{L})$ of class $[R]$.*

Then $\max_{\bar{\Delta}} u$ is attained only on $\overline{O_0 A_0}$.

Suppose the contrary. Let

$$\max_{\bar{\Delta}} u = u(Q), \quad Q \in \bar{\Delta} \setminus \overline{O_0 A_0}.$$

Represent $(\mathcal{L})$ in the form ⁽¹⁾

$$(\beta u_\eta)_\xi + (\alpha u)_\xi + \gamma u = 0$$

and integrate this equation along the segment PQ :

$$\beta(Q)u_\eta(Q) = \left\{ -a(P)u(Q) + \int_{PQ} [u(Q) - u]\gamma_2 d\xi - u(Q) \int_{PQ} \beta c_2 d\xi \right\} + \int_{PQ} [u(Q) - u]\gamma_1 d\xi - u(Q) \int_{PQ} \beta c_1 d\xi; \quad (1)$$

$$\beta(Q)u_\eta(Q) = -\beta(Q)a(Q)u(Q) - \int_{PQ} \gamma u d\xi. \quad (2)$$

By virtue of conditions (C) and (D), respectively, from (1) and (2) we obtain $u_\eta(Q) > 0$, which is impossible and proves the theorem.

Example 1. Denote by $(\mathcal{L}_m^{\lambda\mu})$ the equation $(\mathcal{L})$ in which

$$a = \frac{\sigma_1}{2(\xi + \eta)} + \frac{f(\xi; \eta)}{(-\xi - \eta)^{2\sigma_1 + \lambda}}, \quad b = \frac{\sigma_1}{2(\xi + \eta)} + \frac{g(\xi; \eta)}{(-\xi - \eta)^{2\sigma_1 + \lambda}},$$

$$c = \frac{r(\xi; \eta)}{(-\xi - \eta)^{2\sigma_1 + \mu}};$$

$$0 < m < 2, \quad \lambda < \frac{2-m}{2+m}, \quad \mu < \frac{4-m}{2(2+m)}, \quad \sigma_1 = \frac{m}{2+m},$$

$$f, (\xi + \eta)f_\xi, g, r \in C^{(0)}(\bar{\Delta}).$$

Lemma 1. *If the length of the segment O_0A_0 is sufficiently small, then the maximum principle is valid for the equation $(\mathcal{L}_m^{\lambda\mu})$.*

Indeed, for a sufficiently small length of O_0A_0 the coefficients of $(\mathcal{L}_m^{\lambda\mu})$ satisfy conditions (C)

$$a(P) = \frac{\sigma_1}{2\eta} \left[1 - (-\eta)^\sigma \frac{2}{\sigma_1} f(0; \eta) \right], \quad \sigma = \frac{2-m}{2+m} - \lambda > 0,$$

$$h = -\frac{m(4+m)}{4(2+m)} \frac{1}{(\xi + \eta)^2} [1 - (-\xi - \eta)^\sigma \mathfrak{W}(\xi; \eta)],$$

$$-a(P) - \int_{PQ} \beta |c| d\xi = -\frac{\sigma_1}{2\eta} [1 - (-\eta)^\sigma \Omega(\xi; \eta)],$$

$$\begin{aligned} \mathfrak{W}(\xi; \eta) = & \frac{4(2+m)^2}{m(4+m)} \left[\left(\lambda + \frac{3}{2}\sigma_1 \right) f(\xi; \eta) - \frac{\sigma_1}{2} g(\xi; \eta) - (\xi + \eta) f_\xi(\xi; \eta) + \right. \\ & \left. + (-\xi - \eta)^\sigma f(\xi; \eta) g(\xi; \eta) - (-\xi - \eta)^{1+\lambda-\mu} r(\xi; \eta) \right], \end{aligned}$$

$$\Omega(\xi; \eta) = \frac{2}{\sigma_1} [f(0; \eta) + (-\eta)^{3/2\sigma_1+\lambda} \omega(\xi; \eta)],$$

$$\omega(\xi; \eta) = \int_0^\xi \frac{|r(t; \eta)|}{(-t - \eta)^{3/2\sigma_1+\mu}} \exp \left\{ \int_0^t \frac{g(\tau; \eta)}{(-\tau - \eta)^{2\sigma_1+\lambda}} d\tau \right\} dt.$$

Example 2.

$$\mathfrak{M}_{\lambda\mu}[u] \equiv u_{\xi\eta} + \left(\frac{\lambda}{\xi + \eta} - \frac{\mu}{\xi - \eta} \right) u_\xi + \left(\frac{\lambda}{\xi + \eta} + \frac{\mu}{\xi - \eta} \right) u_\eta = 0. \quad (\mathfrak{M}_{\lambda\mu})$$

Lemma 2. *For the equation $(\mathfrak{M}_{\lambda\mu})$, for any $0 \leq \lambda \leq 1$, $0 \leq \mu \leq 1$, $\lambda + \mu \neq 0$, the maximum principle is valid.*

Indeed, in Δ conditions (C) are satisfied

$$a(P) = \frac{\lambda + \mu}{\eta} < 0; \quad h_1 = -\frac{\lambda - \lambda^2}{(\xi + \eta)^2} \leq 0; \quad h_2 = \frac{\mu - \mu^2}{(\xi - \eta)^2} \geq 0,$$

$$-a(P) - 2 \int_0^\xi |\gamma_2| d\xi > -\frac{\lambda + \mu}{\eta} + \frac{2\mu}{\eta} (1 - 2^{\mu-1}) > 0.$$

It can be shown that for the equation $(\mathfrak{M}_{\lambda\mu})$ with $\lambda = 0$, $0 < \mu < 1$ (the Euler-Poisson-Darboux equation) the well-known maximum principle for hyperbolic equations does not hold ⁽¹⁾, theorem 1).

§ 2. Consider the general generalized F. Tricomi equation

$$T_m[u] \equiv \operatorname{sgn} y \cdot |y|^m u_{xx} + u_{yy} + \frac{M(x; y)}{|y|^\beta} u_x + \frac{N(x; y)}{|y|^\alpha} u_y + \frac{F(x; y)}{|y|^\gamma} u = 0 \quad (T_m)$$

in the domain D , bounded by: 1) a simple Jordan arc Γ , situated in the upper half-plane and resting on the axis $y = 0$ at the points $O(0; 0)$ and $A(a; 0)$, $a > 0$; 2) the characteristics OC and AC . Let D_1 and D_2 be, respectively, the subdomains of ellipticity and hyperbolicity of the equation (T_m) ; OA is the line of transition; $M, N, F \in C^{(0)}(\overline{D_1})$; $M, N, F \in C^{(0)}(\overline{D_2})$; $F \leq 0$ in D_1 ; $M, N \in C^{(1)}(\overline{D_2})$; $N \geq 0$ in D_1 only when $\alpha > 0$; $m \geq 0$, $\beta < 1 - m/2$, $\alpha < 1$, $\gamma < 1 - m/4$. We shall call the function $u(x; y)$ a **regular solution** of the equation $T_m[u] \equiv G(x; y)$ ($G(x; y) \in C^{(0)}(\overline{D})$), if $T_m[u] \equiv G(x; y)$ in $D_1 \cup D_2$; $u \in C^{(2)}(D_1 \cup D_2)$; $u \in C^{(0)}(\overline{D})$; $u \in C^{(1)}[D_1 \cup (D_2 \setminus \overline{OA})]$; $u_y(x; 0+0) = u_y(x; 0-0)$ for $0 < x < a$, and these limits are finite—the gluing condition. A regular solution of the equation (T_m) will be regarded as belonging to the class $[P_0]$ if

$$u|_{\overline{OC}} \equiv 0, \quad \max_{\overline{D}} u > 0, \quad \max_{\overline{D}} u \geq -\min_{\overline{D}} u.$$

Problem T (F. Tricomi). Find a regular solution $u(x, y)$ of the equation $T_m[u] = G(x; y)$ from the data: 1) $u|_{\Gamma} = \varphi$, 2) $u|_{\overline{OC}} = \psi$, where φ and ψ are continuous functions; $\varphi(0) = \psi(0)$.

We shall say that for equation (T_m) the principle of the (strict) maximum for problem T is valid if an arbitrary solution of class $[P_0]$ attains its maximum in $\overline{D}$ (only) in the closure of the “elliptic” arc Γ .

Theorem 2. *If in equation (T_m) , $0 < m < 2$, $\alpha = \beta = \gamma = 0$, and the length of the transition line is sufficiently small, then the principle of the strict maximum for problem T is valid for it.*

Suppose the contrary. Let $u(x; y) \in [P_0]$, $\max_{\overline{D}} u = u(Q)$, but $Q \in \overline{D} \setminus \overline{\Gamma}$. By the known property of elliptic equations and Lemma 1 the point $Q \in OA$. Consequently, by the modified lemma of E. Hopf ⁽¹⁾, Lemma 2, $u_y(x_Q; 0+0) < 0$. But $u_y(x_Q; 0-0) \geq 0$, which contradicts the gluing condition and proves what is required.

Theorem 3. *If in equation (T_m) , $0 < m < 2$, and the length of the transition line is sufficiently small, then the maximum principle for problem T is valid for it.*

Suppose the contrary. Let $u(x; y) \in [P_0]$, $\max_{D_1} u = u(Q)$, but $Q \in \bar{\Gamma}$. Consequently, $Q \in OA$. But then, by virtue of the lemma of K. I. Babenko ⁽²⁾ (one can show that its assertion is also valid in the case $\alpha > 0$, but with $N \geq 0$ in D_1), $u_y(x_Q; 0+0) < 0$, which is impossible by the gluing condition.

Theorem 4. *Let the coefficients of an equation of the form $(\mathcal{L})$, corresponding to the equation (T_m) , satisfy in the domain Δ the conditions (C) or (D) . Then for equation (T_m) the following are valid: 1) the principle of the strict maximum for problem T , if $\alpha = \beta = \gamma = 0$; 2) the maximum principle for problem T .*

The proof is analogous to the proofs of Theorems 2 and 3.

Theorem 5. *For the equation $T_m[u] = G(x; y)$, the solution of problem T is unique if $0 < m < 2$ and the length of the transition line is sufficiently small.*

The proof follows easily from Theorems 2 and 3.

Theorem 6. *If the coefficients of equation (T_m) satisfy the conditions of Theorem 4, then for the equation $T_m[u] = G(x; y)$ the solution of problem T is unique.*

§ 3. Consider an equation of mixed type with two mutually perpendicular parabolicity lines of the first kind

$$T_{mn}[u] \equiv \operatorname{sgn} y \cdot |y|^m U_{xx} + \operatorname{sgn} x \cdot |x|^n u_{yy} = 0, \quad (T_{mn})$$

$$m \geq 0, \quad n \geq 0, \quad m + n \neq 0$$

in a domain D , consisting of three subdomains D_1, D_2, D_3 . The domain $D_1(x > 0; y > 0)$ is bounded by the coordinate axes and by a simple Jordan arc Γ resting on them at the points $A(a; 0)$ and $B(0; b)$. The domains $D_2(x > 0; y < 0)$ and $D_3(x < 0; y > 0)$ are bounded respectively by two pairs of characteristics OC and AC , OE and BE . We shall call a function $u(x; y)$ a regular solution of equation (T_{mn}) if $T_{mn}[u] \equiv 0$ in $D_1 \cup D_2 \cup D_3$, $u \in C^{(2)}(D_1 \cup D_2 \cup D_3)$, $u \in C^{(0)}(\bar{D})$, $u \in C^{(1)}[D_1 \cup (\bar{D}_2 \setminus \overline{OA}) \cup (\bar{D}_3 \setminus \overline{OB})]$,

$$u_y(x; 0+0) = u_y(x; 0-0) \quad \text{for } 0 < x < a, \quad u_x(0+0; y) = u_x(0-0; y)$$

for $0 < y < b$, and these limits are finite—the gluing conditions. We shall regard a regular solution $u(x; y)$ of equation (T_{mn}) as belonging to the class $[P_{00}]$, if

$$u|_{\overline{OC} \cup \overline{OE}} \equiv 0, \quad \max_{\bar{D}} u > 0, \quad \max_{\bar{D}} u \geq -\min_{\bar{D}} u.$$

Problem T. Find a regular solution $u(x; y)$ of equation (T_{mn}) from the data: 1) $u|_{\Gamma} = \varphi$, 2) $u|_{\overline{OC}} = \psi_1$, 3) $u|_{\overline{OE}} = \psi_2$, where φ, ψ_1, ψ_2 are continuous functions; $\psi_1(0) = \psi_2(0)$.

Theorem 7 (the strong maximum principle for problem T). An arbitrary solution of the equation (T_{mn}) of class $[P_{00}]$ attains its maximum in $\bar{D}$ only in the closure of the “elliptic” arc Γ .

The proof is analogous to Theorem 2 and follows from the modified E. Hopf lemma¹ and Lemma 2.

Theorem 8. For the equation (T_{mn}) , the solution of problem T is unique.

The proof follows from Theorem 7.

In conclusion we note that a somewhat different extremum principle for problem T in the case of the Lavrent’ev-Bitsadze and Tricomi equations belongs to A. V. Bitsadze³.

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Note: Figure translations are in progress. See original paper for figures.

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