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Abstract

Full Text

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MATHEMATICS

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ON A PROBLEM IN QUEUEING THEORY AND A RELATED GENERALIZATION OF THE CYLINDRICAL FUNCTIONS J_k

1. It will be convenient for us to use coordinates y_j , $j = 0, 1, \dots, n$, in the n -dimensional Euclidean space E_n , which we shall call **barycentric** (abbreviated b.c.) (in what follows $\sum_j$ and $\prod_j$ denote $\sum_{j=0}^n$, $\prod_{j=0}^n$). Let the unit vectors e_j , $j = 0, 1, \dots, n$, be the vertices of a regular n -dimensional simplex D_n , whose center lies at the origin O . Every vector $Y \subset E_n$ will be represented in the form $Y = \sum_j y_j e_j = [y_j]$, where y_j are b.c. of the vector Y . Since, for any y , $[y_j] = [y_j + y]$, one may fix one of the b.c. or their sum. For $\sum_j y_j = 1$ we obtain the usual b.c. Setting

$$y_{cp} = \frac{1}{n+1} \sum_j y_j,$$

we obtain the **canonical** b.c. y'_j of the vector $[y_i] = [y'_j]$,

$$y'_j = y_j - y_{cp}, \quad \sum_j y'_j = 0.$$

Denoting $\bar{y} = \min_j y_j$, $\bar{y}_j = y_j - \bar{y}$; $\bar{y}_j \geq 0$, $\min \bar{y}_j = 0$, we shall call $\bar{y}_j$ the **second canonical** b.c. of the vector $[y_j] = [\bar{y}_j]$. For $Y = [y_i] = [y'_j]$ we have

$$\begin{aligned} \|Y\|^2 &= \frac{n+1}{n} \sum_j y_j^2 - \frac{1}{n} \left(\sum_j y_j \right)^2 = \frac{n+1}{n} \sum_j y_{j'}^2 = \\ &= \frac{n+1}{n} \sum_{k=1}^n \frac{k}{k+1} \left[y_k - \frac{1}{k} \sum_{i=0}^{k-1} y_i \right]^2. \end{aligned} \quad (1)$$

By Ω_n we shall denote the lattice of integer vectors $K = [k_j]$ in E_n , where k_j are integers; W_n is the Dirichlet region of the lattice Ω_n , its volume

$$|W_n| = \sqrt{(n+1)^{n-1}/n^n}.$$

For a function $f(Y) = f([y_j])$ defined in E_n ,

$$\sum_j \frac{\partial f}{\partial y_j} = 0; \quad \sum_j \frac{\partial^2 f}{\partial y_j^2} = \frac{n+1}{n} \Delta f = \frac{n+1}{n} \sum_{i=1}^n \frac{\partial^2 f}{\partial x_i^2} \quad (2)$$

(x_i are Cartesian coordinates).

2. The assembly problem. A generalization of the problem of taxis and passengers ⁽¹⁾ is the assembly problem. Given are $(n+1)$ streams P_j , $j = 0, 1, \dots, n$. The probability that by time t there arrive k elements of the stream P_j is equal to $p_j(k, t)$. We shall regard the streams as Poisson:

$$p_j(k, t) = e^{-\lambda_j t} \frac{(\lambda_j t)^k}{k!} \quad \left(\frac{1}{k!} = 0 \text{ for } k < 0 \right). \quad (3)$$

Abstracting from the assembly time, we assume that if by time t there have appeared k_j elements of the stream P_j , $j = 0, 1, \dots, n$, then $k = \min_j k_j$ elements of the output stream have been formed— $(n+1)$ -tuples, each of which consists of $n+1$ elements of the streams P_j , $j = 0, 1, \dots, n$, one from each P_j . At the same time queues have formed of $\bar{k}_j = k_j - k$ elements of the stream P_j . All $\bar{k}_j \geq 0$, $\min_j \bar{k}_j = 0$. We shall say: the state vector (k_j) , $j = 0, 1, \dots, n$, of the system has generated the queue vector $(\bar{k}_j)$. Likewise, the queue vector generates vector states $(k_j + m)$, m an arbitrary integer. We form the vector $K = [\bar{k}_j] = [k_j] = [k_j + m] \subset \Omega_n$. The components $\bar{k}_j$ of the vector-queue (k_j) are the second canonical b.c., while the vector states generating it are the various integer b.c. of this vector. Therefore we shall say: a vector-queue $K = [\bar{k}_j] = [k_j]$ has formed; the probabilities $P(K, t)$ of formation at time t of such a queue are equal ($\sum_m = \sum_{m=1}^\infty$):

$$P(K, t) = \sum_m \prod_j p_j(k_j + m, t) = \exp \left(- \sum_j \lambda_j t \right) U_K^{(n)}(\lambda_j t), \quad (4)$$

where

$$\begin{aligned} U_K^{(n)}(x_j) &= U_{(k_j)}^{(n)}(x_j) = U_{k_0, k_1, \dots, k_n}^{(n)}(x_0, x_1, \dots, x_n) = \\ &= U_{(k_j+m)}^{(n)}(x_j) = \sum_m \prod_j \frac{x_j^{m+k_j}}{(m+k_j)!}. \end{aligned} \quad (5)$$

For $x_0 = x_1 = \dots = x_n = x$ we shall have a function of one variable x :

$$U_K^{(n)}(x) = U_{k_0, k_1, \dots, k_n}^{(n)}(x) = \sum_m \prod_j \frac{x^{m+k_j}}{(m+k_j)!} = \sum_m \frac{x^{(n+1)m+k}}{\prod_j (m+k_j)!}, \quad k = \sum_j k_j.$$

For $n = 1$, $U_{k_0, k_1}^{(1)}(x) = J_{k_1 - k_0}(2x)$.

The functions introduced are only loosely connected with Akimov's generalized Bessel functions ⁽²⁾.

3. Generating function. Let $\prod_j t_j = 1$; for $K = [k_j]$, $t^K = \prod_j t_j^{k_j}$; then

$$\exp\left(\sum_j t_j x_j\right) = \sum_{K \subset \Omega_n} t^K U_K^{(n)}(x_j). \quad (6)$$

From this follow the addition theorem (7) and formulas (8)–(10):

$$U_K^{(n)}(x_j + y_j) = \sum_{K' \subset \Omega_n} U_{K'}^{(n)}(x_j) U_{K-K'}^{(n)}(y_j); \quad (7)$$

$$\frac{\partial}{\partial x} U_K^{(n)}(x) = \frac{1}{n+1} U_{K-e_j}^{(n)}(x); \quad (8)$$

$$\sum_j U_{K-e_j}^{(n)}(x_i) x_j e_j = U_K^{(n)}(x_i) K; \quad (9)$$

$$\sum_j x_j e_j = \sum_{K \subset \Omega_n} K U_K^{(n)}(x_i). \quad (10)$$

Let $Y \subset [y_j] = [y'_j]$, $K = [k_j]$; $KY = \sum_j k_j y_j$; $a_K(Y) = \exp(2\pi i KY)$. The system $\{a_K(Y)\}_{K \subset \Omega_n}$ is orthogonal in Y in W_n , complete in the class of functions periodic with periods e_j and with integrable squares on W_n . Every such function $f(Y)$ is representable by the Fourier series

$$f(Y) = \sum_{K \subset \Omega_n} c_K a_K(Y), \quad c_K = \frac{1}{|W_n|} \int_{W_n} \dots \int f(Y) a_K(Y) d\omega_n. \quad (11)$$

Putting in (6) $t_j = \exp(2\pi i y'_j)$, $\sum_j y'_j = 0$, $Y = [y'_j]$, we obtain:

$$\exp\left(\sum_j x_j \exp(2\pi i y'_j)\right) = \sum_{K \subset \Omega_n} U_K(Y) a_K(Y). \quad (12)$$

Hence, from (11) follows the integral representation

$$U_{(k_j)}^{(n)}(x_j) = U_K^{(n)}(x_j) = \frac{1}{|W_n|} \int_{W_n} \cdots \int \exp \left[\sum_j (\exp(2\pi i y'_j) - 2\pi i k_j y_j) \right] d\omega_n. \quad (13)$$

4. Asymptotics. Let $x \gg 1$, $k_j = O(\sqrt{x})$, $K = [k_j] \subset \Omega_n$,

$$U_K^{(n)}(x) = \frac{1}{\sqrt{(n+1)(2\pi x)^n}} \exp \left[-\frac{n\|K\|^2}{2(n+1)x} \right] \left[1 + O\left(\frac{1}{\sqrt{x}}\right) \right]. \quad (14)$$

Let

$$x \gg 1; \quad \lambda_j > 0; \quad z_j = \lambda_j - k_{jx} = O(\sqrt{x});$$

$$S_k = \sum_{i=0}^k \frac{1}{\lambda_i}, \quad k = 0, 1, \dots, n; \quad z^{(k)} = \frac{1}{S_{k-1}} \sum_{i=0}^{k-1} \frac{z_i}{\lambda_i}, \quad k = 1, 2, \dots, n+1;$$

$$\begin{aligned} \Phi(z_j) &= \sum_j \frac{z_j^2}{\lambda_j} - \frac{1}{S_n} \left(\sum_j \frac{z_j}{\lambda_j} \right)^2 \equiv \sum_j \frac{1}{\lambda_j} (z_j - z^{(n+1)})^2 \equiv \\ &\equiv \sum_{k=1}^n \frac{S_{k-1}}{\lambda_k S_k} (z_k - z^{(k)})^2; \end{aligned}$$

$$U_K^{(n)}(z_j) = \left[\prod_j \lambda_j / S_n (2\pi x)^n \right]^{1/2} \exp \left[-\frac{1}{2x} \Phi(z_j) \right] \left[1 + O\left(\frac{1}{\sqrt{x}}\right) \right]; \quad (15)$$

$$O\left(\frac{1}{\sqrt{x}}\right) \sim \sum_{p=1}^{\infty} \frac{a_p}{(\sqrt{x})^p};$$

a_p is a polynomial of degree p in $k_i/\sqrt{x}$ in (14), and in $z_i/\sqrt{x}$ in (15).

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¹ Kh. Sh. Margulis, Collection *Cybernetics in the Service of Communism*, 2, 1964, p. 266. ² M. I. Akimov, *On Bessel Functions of Many Variables and Their Applications in Mechanics*, Petrograd, 1922.

Note: Figure translations are in progress. See original paper for figures.

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