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Abstract

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MATHEMATICS

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ON CONSTRUCTIVE SETS WITH EQUALITY AND THEIR MAPPINGS

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The present note belongs to the constructive trend in mathematics ($\hat{1}$).

The term **set** is used here as a synonym for the term “condition with one parameter” ($\hat{2}$), § 7). If a is a condition with one parameter and x is a constructive object satisfying the condition a , then x is called an **element of the set a** . We shall consider only such sets whose elements are words.

Let a be a set. Suppose that, for the elements of this set, a reflexive, symmetric, and transitive relation has been introduced (by means of a two-place predicate). In this case we shall say that an equality relation has been introduced in the set a , and any two elements of this set that stand in the equality relation will be called **equal**. The pair of objects consisting of the set a and the predicate defining equality of the elements of the set a will be called a **set with equality**, or, briefly, a **p -set**; the set a itself will be called the **base** of this p -set.

Let $\mathcal{A}$ be a p -set. By **elements** of this p -set we shall mean the elements of the base of this p -set. If x and y are elements of the p -set $\mathcal{A}$, then the notation $x = y$ will mean that x and y are equal elements.

The concepts of a set and a p -set are made precise with the aid of some definite logical-mathematical language. The questions considered in this note do not require a compulsory fixation of the language. In those cases where we shall assert the possibility of constructing p -sets satisfying certain conditions, we shall have in mind p -sets for which the base and the condition defining equality of elements are expressible by means of the languages described in §§ 3 and 8 of work ($\hat{2}$).

Let $\mathcal{A}$ and $\mathcal{B}$ be p -sets and let $\mathfrak{A}$ be a normal algorithm ($\hat{3}$). We shall call $\mathfrak{A}$ a **partial mapping** of $\mathcal{A}$ into $\mathcal{B}$ if, for every element x of the p -set $\mathcal{A}$, from $\mathfrak{A}(x)$ it follows that $\mathfrak{A}(x) \in \mathcal{B}$. Let $\mathcal{A}$ and $\mathcal{B}$ be p -sets and let $\mathfrak{A}$ be a partial mapping of $\mathcal{A}$ into $\mathcal{B}$. We shall call $\mathfrak{A}$ a **total mapping** if the algorithm $\mathfrak{A}$

is applicable to every element of the p -set $\mathcal{A}$. We shall call $\mathfrak{A}$ an **equality-preserving mapping** if, for all elements x and y of the p -set $\mathcal{A}$, from $!\mathfrak{A}(x)$, $!\mathfrak{A}(y)$, and $x = y$, it follows that

$$\mathfrak{A}(x) = \mathfrak{A}(y).$$

We shall call $\mathfrak{A}$ a **univalent mapping** if, for any elements x and y of the p -set $\mathcal{A}$ that are not equal, from $!\mathfrak{A}(x)$ and $!\mathfrak{A}(y)$ it follows that the elements $\mathfrak{A}(x)$ and $\mathfrak{A}(y)$ of the p -set $\mathcal{B}$ are likewise not equal. We shall call $\mathfrak{A}$ a **mapping of the p -set $\mathcal{A}$ onto the p -set $\mathcal{B}$** if, for every element x of the p -set $\mathcal{B}$, there cannot fail to exist an element y of the p -set $\mathcal{A}$ such that $!\mathfrak{A}(y)$ and

$$\mathfrak{A}(y) = x.$$

Mappings preserving equality are considered quite often in constructive mathematics. Examples of such mappings are constructive functions of a real variable (in the sense of A. A. Markov (³)); mappings preserving equality and satis-

satisfying certain additional conditions are homomorphisms and isomorphisms of associative calculi (³) (with the equivalence relation of words in the corresponding associative calculi taken as the equality relation of words), etc.

In the present note a classification of p -sets is described according to certain properties of their bases and of the conditions determining equality of elements, and some results are given concerning mappings of p -sets that preserve equality. These results are formulated in terms of the classification of p -sets described here.

Let $\mathcal{A}$ be a p -set. We shall say that $\mathcal{A}$ is **decidable** if the basis of $\mathcal{A}$ is a decidable set. We shall say that $\mathcal{A}$ is **enumerable** if the basis of $\mathcal{A}$ is an enumerable set. We shall call $\mathcal{A}$ **conditionally enumerable** if there exists an enumerable set β such that every element of the set β is an element of the p -set $\mathcal{A}$, and for every element x of the p -set $\mathcal{A}$ there cannot fail to exist an element y of this p -set such that $y \in \beta$ and $x = y$. We shall say that $\mathcal{A}$ has a **decidable equality condition** if there exists a normal algorithm applicable to every pair of elements of the p -set $\mathcal{A}$ and recognizing pairs consisting of equal elements. If x is an element of the p -set $\mathcal{A}$, then by $R(x)$ we shall denote a set such that: a) every element of the set $R(x)$ is an element of the p -set $\mathcal{A}$ equal to the element x ; b) every element of the p -set $\mathcal{A}$ equal to the element x is an element of the set $R(x)$. We shall say that the p -set $\mathcal{A}$ has a **completely enumerable equality condition** if for every element x of this p -set the set $R(x)$ is enumerable. We shall say that the p -set $\mathcal{A}$ has a **normal equality condition** if, for all elements x and y of this p -set, the judgment “ x and y are equal” is equivalent to its double negation.

We shall say that the p -set $\mathcal{A}$ is **infinite** if there does not exist a finite collection of elements of this p -set that would satisfy the condition: for every element x

of the p -set $\mathcal{A}$ there cannot fail to exist an element equal to x and belonging to this collection. Here we shall confine ourselves to considering only infinite p -sets.

By **types** of p -sets we shall mean words of the form $[a, b]$, where as a one takes the letter P, Π , H, or the word $\sim$, and as b one takes one of the symbols $P =$, $\Pi =$, $\Pi \neq$, $\sim =$, or $H =$.

Let $\mathcal{A}$ be a p -set. We shall say that $\mathcal{A}$ **has type** $[a, b]$ if $\mathcal{A}$ is decidable and a is P, or $\mathcal{A}$ is enumerable and a is Π , or $\mathcal{A}$ is conditionally enumerable and a is $\sim$, or a is H and it is false that $\mathcal{A}$ is conditionally enumerable, and if $\mathcal{A}$ has a decidable equality condition and b is $P =$, or $\mathcal{A}$ has an enumerable equality condition ⁽⁵⁾ and b is $\Pi =$, or $\mathcal{A}$ has an enumerable inequality condition ⁽⁵⁾ and b is $\Pi \neq$, or $\mathcal{A}$ has a completely enumerable equality condition and b is $\sim =$, or b is $H =$ and it is false that $\mathcal{A}$ has at least one of the just-mentioned equality or inequality conditions.

The same p -set may have several different types. If T and U are types of p -sets, then by $T \rightarrow U$ we shall denote the judgment: “every infinite p -set having type T also has type U ,” and by $T \sim U$ we shall denote the judgment: “every infinite p -set having one of the types T, U also has the other of these types.” The following relations between types of infinite p -sets hold (here, in place of a and b , one may substitute any symbols allowed for them according to the definition of types of p -sets):

$$[P, b] \rightarrow [\Pi, b], \quad [\Pi, b] \rightarrow [\sim, b], \quad [a, P =] \rightarrow [a, \Pi =],$$

$$[a, P =] \rightarrow [a, \Pi \neq], \quad [a, \sim =] \rightarrow [a, \Pi =],$$

$$[P, \Pi =] \sim [P, \sim =], \quad [\Pi, \Pi =] \sim [\Pi, \sim =] \sim [\sim, \sim =].$$

Considering concrete examples of p -sets, one can show that no other relations of the indicated kind between the types of infinite p -sets exist.

Theorem 1. Let $\mathcal{A}$ and $\mathcal{B}$ be p -sets, and let $\mathcal{A}$ be conditionally enumerable. If there exists a total mapping of $\mathcal{A}$ onto $\mathcal{B}$ that preserves equality, then $\mathcal{B}$ is also conditionally enumerable.

Theorem 2. Let $\mathcal{A}$ and $\mathcal{B}$ be infinite p -sets, and let $\mathcal{A}$ have type $[\sim, P =]$, while $\mathcal{B}$ is conditionally enumerable. Then one can construct a total mapping of $\mathcal{A}$ onto $\mathcal{B}$ that preserves equality. If the p -set $\mathcal{B}$ also has type $[\sim, P =]$, then such a mapping can be constructed one-to-one.

Let $\mathcal{A}$ and $\mathcal{B}$ be p -sets and let $\mathfrak{A}$ be a partial mapping of $\mathcal{A}$ into $\mathcal{B}$ preserving equality. We shall call $\mathfrak{A}$ a **pretotal mapping** if, for every element x of the p -set $\mathcal{A}$, there cannot fail to exist an element of this p -set equal to it to which

the algorithm $\mathfrak{A}$ is applicable. We shall say that $\mathfrak{A}$ is **extendable to a total mapping preserving equality** if one can construct a total mapping $\mathfrak{B}$ of the p -set $\mathcal{A}$ into the p -set $\mathcal{B}$, preserving equality, such that for every element x of the p -set $\mathcal{A}$, if $\mathfrak{A}(x)$, then $\mathfrak{A}(x) = \mathfrak{B}(x)$.

Theorem 3. Let $\mathcal{A}$ be a p -set having a completely enumerable equality condition. Then every pretotal mapping of the p -set $\mathcal{A}$ into any p -set $\mathcal{B}$, preserving equality, is extendable to a total mapping preserving equality.

Let $\mathcal{A}$ and $\mathcal{B}$ be p -sets and let $\mathfrak{A}$ be a partial mapping of $\mathcal{A}$ into $\mathcal{B}$. By a mapping **inverse to $\mathfrak{A}$** we shall mean any partial mapping $\mathfrak{B}$ of the p -set $\mathcal{B}$ into the p -set $\mathcal{A}$ such that, for every element x of the p -set $\mathcal{B}$: a) if $\mathfrak{B}(x)$, then $\mathfrak{A}(\mathfrak{B}(x))$ and $\mathfrak{A}(\mathfrak{B}(x)) =_B x$; b) if there exists an element y of the p -set $\mathcal{A}$ such that $\mathfrak{A}(y)$ and $\mathfrak{A}(y) =_B x$, then $\mathfrak{B}(x)$. Let us note that an algorithm inverse to the algorithm $\mathfrak{A}$ in the sense defined in (6) is not always a mapping inverse to $\mathfrak{A}$.

Theorem 4. Let $\mathcal{A}$ be a conditionally enumerable p -set having a normal equality condition, and let $\mathcal{B}$ be a p -set having an enumerable equality condition. Then, for every total one-to-one mapping $\mathfrak{A}$ of the p -set $\mathcal{A}$ into the p -set $\mathcal{B}$, preserving equality, one can construct a mapping inverse to $\mathfrak{A}$ and preserving equality.

Theorem 5. Let $\mathcal{A}$ be a conditionally enumerable p -set and let $\mathcal{B}$ be a p -set having a decidable equality condition. Then, for every total mapping $\mathfrak{A}$ of the p -set $\mathcal{A}$ into the p -set $\mathcal{B}$, preserving equality, one can construct a mapping inverse to $\mathfrak{A}$ and preserving equality.

Let $\mathcal{A}$ and $\mathcal{B}$ be p -sets. We shall say that $\mathcal{A}$ is **superposable on $\mathcal{B}$** if there exists a total one-to-one mapping of $\mathcal{A}$ onto $\mathcal{B}$ preserving equality. We shall say that $\mathcal{A}$ and $\mathcal{B}$ are **equivalent** if there exists a total mapping $\mathfrak{A}$ of the p -set $\mathcal{A}$ onto the p -set $\mathcal{B}$, and there exists a total mapping $\mathfrak{B}$ of the p -set $\mathcal{B}$ onto the p -set $\mathcal{A}$, both one-to-one and preserving equality, such that $\mathfrak{A}$ is a mapping inverse to $\mathfrak{B}$, and $\mathfrak{B}$ is a mapping inverse to $\mathfrak{A}$.

One can construct an example of p -sets $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{A}$ is superposable on $\mathcal{B}$, $\mathcal{B}$ is superposable on $\mathcal{A}$, but $\mathcal{A}$ and $\mathcal{B}$ are not equivalent.

By the **principal types** of p -sets we shall mean types $[a, b]$, where a is taken to be $\neq$ or $=$, and b is taken to be $P =$, $=$, $\neq$, or $=$. One and the same p -set may have several different principal types.

Theorem 6. Let $\mathcal{A}$ and $\mathcal{B}$ be p -sets and let T be a principal type. If $\mathcal{A}$ and $\mathcal{B}$ are equivalent and $\mathcal{A}$ has type T , then $\mathcal{B}$ has type T .

Theorem 7. Any two infinite p -sets having type $[UP, P =]$ are equivalent.

For every principal type T of p -sets distinct from the type $[UP, P =]$, one can construct p -sets having type T and not equivalent.

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