

# HYDRODYNAMIC EQUATIONS

A. V. Svidzinskii, V. A. Slyusarev

1967

SovietRxiv

---

View the original and related papers at <https://sovietrxiv.org/items/ru-196701.27048>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

## Abstract

## Full Text

UDC 537.312.62

## PHYSICS

A. V. SVIDZINSKII, V. A. SLYUSAREV

# HYDRODYNAMIC EQUATIONS

# IN THE THEORY OF SUPERCONDUCTIVITY

(Presented by Academician N. N. Bogolyubov, 23 III 1966)

The purpose of this article is to derive the equations of two-fluid hydrodynamics for London superconductors from the equations of the microscopic theory of superconductivity.

In the nonequilibrium case a superconductor can be described by the following equations for the correlation functions

$$\begin{aligned} & \langle \psi_{\uparrow}^+(x_1) \psi_{\uparrow}(x_2) \rangle \quad \text{and} \quad \langle \psi_{\downarrow}(x_1) \psi_{\uparrow}(x_2) \rangle : \\ & \left( i\hbar \frac{\partial}{\partial t} - eA_0(x_2) + eA_0(x_1) \right) \langle \psi_{\uparrow}^+(x_1) \psi_{\uparrow}(x_2) \rangle = \\ & = -\frac{1}{2m} \left[ \left( \frac{\hbar}{i} \nabla_2 - \frac{e}{c} \mathbf{A}(x_2) \right)^2 - \left( \frac{\hbar}{i} \nabla_1 + \frac{e}{c} \mathbf{A}(x_1) \right)^2 \right] \langle \psi_{\uparrow}^+(x_1) \psi_{\uparrow}(x_2) \rangle - \\ & - \Delta(x_2) \langle \psi_{\uparrow}^+(x_1) \psi_{\downarrow}^+(x_2) \rangle + \Delta^*(x_1) \langle \psi_{\downarrow}(x_1) \psi_{\uparrow}(x_2) \rangle, \end{aligned} \quad (1)$$

$$\begin{aligned} & \left( i\hbar \frac{\partial}{\partial t} - eA_0(x_1) - eA_0(x_2) \right) \langle \psi_{\downarrow}(x_1) \psi_{\uparrow}(x_2) \rangle = \\ & = -\frac{1}{2m} \left[ \left( \frac{\hbar}{i} \nabla_1 - \frac{e}{c} \mathbf{A}(x_1) \right)^2 + \left( \frac{\hbar}{i} \nabla_2 - \frac{e}{c} \mathbf{A}(x_2) \right)^2 \right] \langle \psi_{\downarrow}(x_1) \psi_{\uparrow}(x_2) \rangle - \\ & - \Delta(x_1) \delta(\mathbf{r}_1 - \mathbf{r}_2) + \Delta(x_1) \langle \psi_{\uparrow}^+(x_1) \psi_{\uparrow}(x_2) \rangle + \Delta(x_2) \langle \psi_{\uparrow}^+(x_2) \psi_{\downarrow}(x_1) \rangle. \end{aligned} \quad (2)$$

Here  $\Delta(x) = g\langle\psi\downarrow(x)\psi\uparrow(x)\rangle$ ;  $x_{1,2} = (\mathbf{r}_{1,2}, t)$ ;  $\mathbf{A}$  is the vector potential,  $A_0$  the scalar potential. The averaging is understood over a nonequilibrium ensemble. In deriving (1)–(2), the generalized Hartree–Fock decoupling <sup>(1,2)</sup> was used. Terms not essentially connected with the phenomenon of superconductivity were omitted.

We emphasize that within the approximation made it is impossible to take relaxation effects into account, and therefore we can obtain the hydrodynamic equations only in the inviscid approximation.

We separate out the gradient-noninvariant factors by putting

$$\begin{aligned}\langle\psi_{\sigma}^{+}(x_1)\psi_{\sigma}(x_2)\rangle &= \exp\left[\frac{im}{\hbar}(\chi(x_2) - \chi(x_1))\right] \Phi_{\sigma}(x_1, x_2), \\ \langle\psi_{\downarrow}(x_1)\psi_{\uparrow}(x_2)\rangle &= \exp\left[\frac{im}{\hbar}(\chi(x_1) - \chi(x_2))\right] \psi^{*}(x_1, x_2), \\ \Delta(x) &= |\Delta(x)| \exp\left[\frac{2im}{\hbar}\chi(x)\right].\end{aligned}\quad (3)$$

The equations for  $\Phi$  and  $\psi$  will contain explicitly the gradient-invariant combinations of the potentials and phase. Put, by definition,

$$\mathbf{v}_s = \nabla\chi - \frac{e}{mc}\mathbf{A}.\quad (4)$$

Let us pass to the mixed representation for the correlation functions  $\Phi$  and  $\psi$ , in which their arguments will be  $\mathbf{R} = \frac{1}{2}(\mathbf{r}_1 + \mathbf{r}_2)$  and  $\mathbf{p}$ —the momentum conjugate to  $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ . Since  $\frac{\hbar}{i}\nabla_1 \rightarrow \mathbf{p} - \frac{i\hbar}{2}\nabla_R$ ,  $\mathbf{r}_1 \rightarrow \mathbf{R} +$

$+i\hbar/2\nabla_p$ , then the expansion in gradients formally coincides with the quasi-classical expansion. To obtain the hydrodynamic equations it is sufficient to consider the zeroth and first approximations in  $\hbar$ . In the zeroth approximation, from (1)–(4) we have

$$\psi_0^{*}(\mathbf{R}, \mathbf{p}, t) = \psi_0(\mathbf{R}, \mathbf{p}, t),\quad (5)$$

$$\begin{aligned}&\left(eA_0(\mathbf{R}, t) + m\dot{\chi}(\mathbf{R}, t) + \frac{mv_s^2}{2}\right)\psi_0 = \\ &= -\frac{p^2}{2m}\psi_0 + \frac{1}{2}|\Delta_0(\mathbf{R}, t)|(1 - \Phi_{\uparrow 0}(\mathbf{R}, \mathbf{p}, t) - \Phi_{\downarrow 0}(\mathbf{R}, -\mathbf{p}, t)).\end{aligned}$$

Equation (5) can be reduced to two equations by introducing the separation parameter  $\mu$ . Denoting  $\xi = p^2/2m - \mu$ , we have

$$eA_0(\mathbf{R}, t) + m\dot{\chi}(\mathbf{R}, t) + \mu(\mathbf{R}, t) + mv_s^2/2 = 0, \quad (6)$$

$$2\xi\psi_0(\mathbf{R}, \mathbf{p}, t) - |\Delta_0(\mathbf{R}, t)|(1 - \Phi_{\uparrow 0}(\mathbf{R}, \mathbf{p}, t) - \Phi_{\downarrow 0}(\mathbf{R}, -\mathbf{p}, t)) = 0. \quad (7)$$

Equation (7) is brought to the usual form of the equation for the order parameter if  $\mu$  is identified with the chemical potential and the locally equilibrium expressions for  $\psi_0$  and  $\Phi_0$  are substituted. Applying the operator  $\nabla_R$  to (6), we obtain the equation for the superfluid velocity

$$\frac{\partial \mathbf{v}_s}{\partial t} + \nabla \left( \frac{v_s^2}{2} + \frac{\mu}{m} \right) = \frac{e}{m} \mathbf{E}. \quad (8)$$

To obtain the equations determining the time evolution of the other hydrodynamic quantities, it is necessary to turn to the equation of the first approximation in  $\hbar$

$$\begin{aligned} & \frac{\partial \Phi_{\uparrow 0}(\mathbf{R}, \mathbf{p}, t)}{\partial t} - \frac{p_i + mv_{si}}{m} \frac{\partial \Phi_{\uparrow 0}}{\partial R_i} + \frac{\partial \Phi_{\uparrow 0}}{\partial p_i} \left\{ eE_i - \frac{\partial}{\partial t} (p_i + mv_{si}) - \right. \\ & \left. - \frac{\partial}{\partial R_i} \frac{(\mathbf{p} + m\mathbf{v}_s)^2}{2} \right\} + \frac{\partial |\Delta_0|}{\partial R_i} \frac{\partial \psi_0(\mathbf{R}, \mathbf{p}, t)}{\partial t} = -2|\Delta_0(\mathbf{R}, t)| \operatorname{Im} \psi_1(\mathbf{R}, \mathbf{p}, t), \quad (9) \end{aligned}$$

$$\begin{aligned} & \frac{\partial \psi_0(\mathbf{R}, \mathbf{p}, t)}{\partial t} + 2i\xi\psi_1^* = -\frac{\partial}{\partial R_i} (v_{si}\psi_0(\mathbf{R}, \mathbf{p}, t)) + \\ & + \frac{\partial}{\partial p_j} \left( p_i \frac{\partial v_{si}}{\partial R_j} \psi_0 \right) - i|\Delta_0(\mathbf{R}, t)|(\Phi_{\uparrow 1}(\mathbf{R}, \mathbf{p}, t) + \Phi_{\downarrow 1}(\mathbf{R}, -\mathbf{p}, t)) + \\ & + \frac{\partial |\Delta_0(\mathbf{R}, t)|}{\partial R_i} \frac{\partial}{\partial p_i} (\Phi_{\downarrow 0}(\mathbf{R}, -\mathbf{p}, t) - \Phi_{\uparrow 0}(\mathbf{R}, \mathbf{p}, t)) + \\ & + 2i\xi\psi_0(\mathbf{R}, \mathbf{p}, t) \left| \frac{\Delta_1(\mathbf{R}, t)}{\Delta_0(\mathbf{R}, t)} \right|. \quad (10) \end{aligned}$$

From (9) and (10) we can obtain the hydrodynamic equations. Thus, defining the mass density by the relation

$$\rho(\mathbf{R}, t) = 2m \int \Phi_{\uparrow 0}(\mathbf{R}, \mathbf{p}, t) \frac{d\mathbf{p}}{(2\pi\hbar)^3}, \quad (11)$$

we obtain

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \mathbf{j} = 0, \quad (12)$$

where

$$\mathbf{j} = \mathbf{j}_0 + \rho \mathbf{v}_s, \quad \mathbf{j}_0 = 2 \int \mathbf{p} \Phi_{\uparrow 0} \frac{d\mathbf{p}}{(2\pi\hbar)^3}. \quad (13)$$

In an analogous manner we find

$$\frac{\partial j_k}{\partial t} + \frac{\partial \Pi_{ik}}{\partial R_i} = \frac{e}{m} \rho E_k + \frac{e}{mc} H_{ki} j_i, \quad (14)$$

where  $H_{ki}$  is the tensor of magnetic-field strength;  $\Pi_{ik}$  is the tensor of density of the momentum flux density

$$\Pi_{ik} = \rho v_{si} v_{sk} + j_{0i} v_{sk} + j_{0k} v_{si} + \Pi_{ik,0}, \quad (15)$$

$$\Pi_{ik,0} = 2 \int \frac{p_i p_k}{m} \Phi_{\uparrow 0} \frac{dp}{(2\pi\hbar)^3} - \delta_{ik} \frac{|\Delta_0|^2}{g}. \quad (16)$$

It is somewhat more difficult to establish the continuity equation for the energy, which is, by definition,

$$\mathcal{E} = \mathcal{E}_0 + \mathbf{j}_0 \mathbf{v}_s + \frac{\rho v_s^2}{2}, \quad \mathcal{E}_0 = 2 \int \frac{p^2}{2m} \Phi_{\uparrow 0} \frac{dp}{(2\pi\hbar)^3} - \frac{|\Delta_0|^2}{g}. \quad (17)$$

Using (9), we find for  $\partial \mathcal{E} / \partial t$  the expression

$$\begin{aligned} \frac{\partial \mathcal{E}}{\partial t} = & \frac{e}{m} \mathbf{E} \mathbf{j} - \frac{\partial}{\partial R_i} \int \frac{p_i + m v_{si}}{m} \frac{(\mathbf{p} + m \mathbf{v}_s)^2}{m} \Phi_{\uparrow 0} \frac{dp}{(2\pi\hbar)^3} + \\ & + 2 \frac{\partial |\Delta_0|}{\partial R_i} \int \frac{p_i + m v_{si}}{m} \psi_0 \frac{dp}{(2\pi\hbar)^3} - \frac{2|\Delta_0|}{g} \frac{\partial |\Delta_0|}{\partial t} - 4|\Delta_0| \int \xi \operatorname{Im} \psi_1 \frac{dp}{(2\pi\hbar)^3}. \end{aligned} \quad (18)$$

Taking the real part of equation (10) and using (6), as a result of integration over momenta we obtain

$$\frac{\partial |\Delta_0|}{\partial t} + \operatorname{div} |\Delta_0| \mathbf{v}_s + 2g \int \xi \operatorname{Im} \psi_1 \frac{dp}{(2\pi\hbar)^3} = 0. \quad (19)$$

With the aid of (19) we reduce equation (18) to the form

$$\frac{\partial \mathcal{E}}{\partial t} + \operatorname{div} \mathbf{Q} = \frac{e}{m} \mathbf{E} \mathbf{j}, \quad (20)$$

in which the energy-flux density  $\mathbf{Q}$  is defined by the expression

$$\mathbf{Q} = \left( \mathcal{E}_0 + \mathbf{j}_0 \mathbf{v}_s + \frac{\rho v_s^2}{2} \right) \mathbf{v}_s + \frac{v_s^2}{2} \mathbf{j}_0 + (\Pi_0 \mathbf{v}_s) + \mathbf{Q}_0, \quad (21)$$

$$\mathbf{Q}_0 = 2 \int \frac{p^2}{2m} \frac{\mathbf{p}}{m} \Phi_{\uparrow 0} \frac{dp}{(2\pi\hbar)^3}. \quad (22)$$

The quantities  $\mathbf{j}_0$ ,  $\Pi_{ik,0}$ ,  $\mathbf{Q}_0$  can be found if the locally equilibrium expression is used for  $\Phi_{\uparrow 0}$ . We obtain  $\mathbf{j}_0 = \rho_n \mathbf{u}$ , where the density of the normal component  $\rho_n$  is expressed in terms of the number density of quasiparticles  $f$  with the superconducting dispersion law  $E_p = \sqrt{\xi^2 + |\Delta|^2}$

$$\rho_n = 2 \int \frac{\mathbf{p} \mathbf{u}}{u^2} f \left( \frac{E - \mathbf{p} \mathbf{u}}{\theta} \right) \frac{dp}{(2\pi\hbar)^3}. \quad (23)$$

Thus,

$$\mathbf{j} = \rho_s \mathbf{v}_s + \rho_n \mathbf{v}_n, \quad \rho_s = \rho - \rho_n, \quad \mathbf{u} = \mathbf{v}_n - \mathbf{v}_s. \quad (24)$$

After several cumbersome transformations we find

$$\Pi_{ik} = \rho_n v_{ni} v_{nk} + \rho_s v_{si} v_{sk} + \delta_{ik} P, \quad (25)$$

$$\mathbf{Q} = \left( \frac{v_s^2}{2} + \frac{\mu}{m} \right) \mathbf{j} + \theta S \mathbf{v}_n + \rho_n \mathbf{v}_n (\mathbf{v}_n, (\mathbf{v}_n - \mathbf{v}_s)), \quad (26)$$

where  $S$  is the entropy,  $P$  the pressure,

$$P = \theta S - \mathcal{E}_0 + \mu \rho / m + \rho_n u^2. \quad (27)$$

Equations (4), (12), (14), (20), together with relations (23)–(26), form a complete system of equations of two-fluid hydrodynamics

L. D. Landau<sup>(3)</sup>, generalized to the case of charged particles. Instead of equation (20) we may use the entropy-transport equation

$$\partial S / \partial t + \operatorname{div} S \mathbf{v}_n = 0, \quad (28)$$

which follows from the equations of hydrodynamics and the fundamental thermodynamic identity

$$d\mathcal{E}_0 = \theta dS + \frac{\mu}{m} d\rho + \mathbf{u} d\mathbf{j}_0. \quad (29)$$

In conclusion, we note that the microscopic method for deriving the equations of two-fluid hydrodynamics was first developed for the Bose-system example by N. N. Bogolyubov<sup>(3)</sup>. This method admits a direct extension to the case of Fermi systems, which was carried out in Ref. <sup>(5)</sup> for He<sup>3</sup>. Our method differs from that used in these works. Simultaneously with the publication of our work as a VINITI preprint<sup>(6)</sup>, an article appeared<sup>(7)</sup> in which the same problem is solved. It should be noted that the method used in <sup>(7)</sup> for solving the basic equations leads at intermediate stages to inconsistent expressions, which, however, does not affect the final result.

Physical-Technical Institute of Low Temperatures  
Academy of Sciences of the Ukrainian SSR

Received  
20 III 1966

## CITED LITERATURE

- <sup>1</sup> N. N. Bogolyubov, *UFN*, **67**, 549 (1959).
- <sup>2</sup> L. P. Gor'kov, *ZhETF*, **34**, 735 (1958).
- <sup>3</sup> L. D. Landau, E. M. Lifshitz, *Mechanics of Continuous Media*, Moscow, 1953.
- <sup>4</sup> N. N. Bogolyubov, *On the Hydrodynamics of a Superfluid Liquid*, Preprint of the Joint Institute for Nuclear Research, R-1395, Dubna, 1963.
- <sup>5</sup> V. Galasiewicz, *Equations of Hydrodynamics of a Superfluid Fermi Liquid and Two-Particle Functions*, Preprint of the Joint Institute for Nuclear Research, R-1953, Dubna, 1965.
- <sup>6</sup> A. V. Svidzinsky, V. A. Slyusarev, *Hydrodynamic Equations in the Microscopic Theory of Superfluid Fermi Systems*, Kharkov, 1965.
- <sup>7</sup> M. J. Stephen, *Phys. Rev.*, **139**, A197 (1965).

*Note: Figure translations are in progress. See original paper for figures.*

*Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.*