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1967

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Abstract

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UDC 513.88 + 517.948.35 + 517.948.5

MATHEMATICS

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UNICELLULARITY AND NONUNICELLULARITY OF WEIGHTED SHIFT OPERATORS

(Presented by Academician V. I. Smirnov on 15 III 1966)

Let $\{H_i\}_{i=0}^{\infty}$ be a sequence of Hilbert spaces; $l^2\{H_i\}_0^{\infty}$ the space of sequences X , $X = \{X_i\}_{i=0}^{\infty}$, $X_i \in H_i$, $i \geq 0$, with finite norm

$$\|X\| = \left(\sum_0^{\infty} |X_i|^2 \right)^{1/2},$$

where $|X_i|$ is the norm of the element X_i in the space H_i , $i = 0, 1, \dots$. If $H_0 = H_1 = \dots = H$, then we write $l^2(H) = l^2\{H_i\}_0^{\infty}$. Let $H_0 \subset H_1 \subset \dots$. Define in the space $l^2\{H_i\}_0^{\infty}$ the shift operator S by the equalities $SX = Y$, $X \in l^2\{H_i\}_0^{\infty}$, where $X = \{X_i\}_0^{\infty}$, $Y = \{Y_i\}_0^{\infty}$ and $Y_0 = 0$, $Y_i = X_{i-1}$, $i \geq 1$, and the operator Λ of multiplication by a (bounded) sequence of complex numbers $\{\lambda_i\}_0^{\infty}$ by the equalities $\Lambda X = Y$, $X \in l^2\{H_i\}_0^{\infty}$, where $X = \{X_i\}_0^{\infty}$, $Y = \{Y_i\}_0^{\infty}$ and $Y_i = \lambda_i X_i$, $i \geq 0$. Finally, let C^q be the q -dimensional complex space and $l^2 = l^2(C^1)$.

Consider in the space $l^2\{H_i\}_0^{\infty}$ the operator $T = S\Lambda$ (operators of this kind are naturally called weighted shift operators). Of known interest is a complete description of the invariant subspaces* of the operators $S\Lambda$ in the spaces $l^2\{H_i\}_0^{\infty}$ (see also ⁽⁶⁾). Beginning with the work of A. Beurling ⁽²⁾ on the operator S in the space l^2 , constant attempts have been made in this direction (see, for example, ^(1,3-5,7)). All these works (except ^(1,3)) adjoin A. Beurling's investigations. In the present note the study, begun in ⁽¹⁾, of invariant subspaces of the operators $S\Lambda$ in the spaces l^2 , $l^2(C^q)$ (and also in "two-sided" sequence spaces, see no. 4) is continued. In particular, it is established that there exist nonunicellular (see the definition below) operators $S\Lambda$, acting in l^2 , with a sufficiently complicated structure of invariant subspaces (including ones such that $\lambda_k \neq 0$, $k \geq 0$, $\lim_k \lambda_k = 0$).

1. Unicellularity of the operators $S\Lambda$ in the space l^2 .

Denote by P_k ($k = 0, 1, \dots$) the operator acting from $l^2\{H_i\}_0^\infty$ into H_k according to the rule $P_{kX} = X_k$, if $X \in l^2\{H_i\}_0^\infty$, $X = \{X_i\}_0^\infty$. Recall that an operator T , acting in some Hilbert space $\mathcal{H}$, is called **unicellular** if, for any two of its invariant subspaces H and H' , at least one of the inclusions $H \subset H'$, $H' \subset H$ holds. For the operators $T = S\Lambda$ acting in l^2 , unicellularity means that every T -invariant subspace coincides with one of the subspaces

$$l_n^2 = \{X : X \in l^2, P_{kX} = 0, \text{ if } k < n\}, \quad n = 0, 1, \dots$$

Denote, as usual, by l^p , $1 \leq p < \infty$, the space of sequences of complex numbers summable to the power p .

Theorem 1. Consider in the space l^2 an operator $T = S\Lambda$ such that $\lambda_i \neq 0$, $i \geq 0$. Let

$$r_i^{(k)} = |\lambda_i \cdots \lambda_{i+k}|, \quad i = 0, 1, \dots; \quad k = 0, 1, \dots,$$

and suppose that the following conditions are fulfilled:

* Here and below, subspaces are always assumed to be closed.

- 1) There exists an integer k , $k \geq 0$, such that $r_i^{(k)} \downarrow 0$, $i \rightarrow \infty$ (i.e. $\lim_i r_i^{(k)} = 0$, $r_0^{(k)} \geq r_1^{(k)} \geq \dots$).
- 2) There exists an integer n , $n \geq 0$, such that $\{r_i^{(n)}\}_{i=0}^\infty \in \bigcup_{p < \infty} l^p$.

Then the operator T is unicellular.

Remark 1. Conditions 1) and 2) of the theorem mean that the sequence $\{\lambda_i\}_0^\infty$ splits into a finite number of subsequences of the form $\{\lambda_{i+kq}\}_{k=0}^\infty$, $i = 0, 1, \dots, q-1$, in each of which $|\lambda_{i+kq}|$ is nonincreasing as $k \rightarrow \infty$, and, moreover,

$$\{\lambda_{kq}\lambda_{1+kq} \cdots \lambda_{q-1+kq}\}_{k=0}^\infty \in \bigcup_{p < \infty} l^p.$$

Example. $\lambda_{2k} = 1$, $k = 0, 1, \dots$; $\lambda_{2k+1} = 1/(2k+1)$, $k = 0, 1, \dots$

Remark 2. Theorem 1 remains valid if the space l^2 is replaced by l^p , $1 < p < \infty$. If $p = 1$, then it suffices to restrict oneself only to requirement 1) in the conditions of Theorem 1 in order to obtain unicellularity of the operator T .

2. Invariant subspaces of the operators $S\Lambda$ in $l^2(C^q)$. Consider the operator $T = S\Lambda$ in the space $l^2(H)$, and let

$$T_1 = \sum_{k=0}^{\infty} \hat{\varphi}_k T^k,$$

where $\hat{\varphi}_k$, $k = 0, 1, \dots$, are operators in $l^2(H)$ such that $\hat{\varphi}_k(\{X_i\}_0^\infty) = \{\varphi_k X_i\}_0^\infty$, and φ_k are bounded operators in H . We shall say of operators T_1 with $\{\|\hat{\varphi}_k\| \cdot |\lambda_0 \dots \lambda_{k-1}|\}_1^\infty \in l^2$ that they are l^2 -functions of the operator T (and in this case the series $\sum_0^\infty \hat{\varphi}_k T^k$ converges in the operator norm), and we shall write $T_1 = \varphi(T)$.

Let us introduce one more notation. Let Y be a subset of the Hilbert space $\mathcal{H}$; let Γ be an operator acting from Y into $\mathcal{H}$ such that ΓY is orthogonal to Y . Put $Y \dot{+} \Gamma Y = \{X + \Gamma X : X \in Y\}$.

Theorem 2. Let $T = S\Lambda$ be an operator acting in $l^2(C^q)$, and let the sequence $\{\lambda_k\}_0^\infty$ satisfy all the conditions of Theorem 1. Let M , $M \subset l^2(C^q)$, be an invariant subspace of the operator T : $TM \subset M$. Then there exist: an integer k , $k \geq 0$, subspaces B_i , $i \geq 0$,

$$B_0 \subset B_1 \subset \dots \subset B_k = B_{k+1} = \dots, \quad B_{k-1} \neq B_k, \quad B_i \subset C^q, \quad i \geq 0,$$

and an l^2 -function $\varphi(T)$ of the operator T , with $\hat{\varphi}_0 = 0$, mapping $l^2\{B_i\}_0^\infty$ into $l^2\{C^q \ominus B_i\}_0^\infty$, such that

$$M = l^2\{B_i\}_0^\infty \dot{+} \varphi(T)l^2\{B_i\}_0^\infty. \quad (1)$$

Under the assumptions made, the triple k , $\{B_i\}_0^\infty$, $\varphi(T)$ is uniquely determined by the subspace M . Conversely, every subspace of the form (1) is invariant with respect to the operator T .

Corollary. Every T -invariant subspace M , $M \subset l^2(C^q)$, is generated by no more than q of its elements (i.e. there are X^i , $1 \leq i \leq q$, $X^i \in M$, such that the closed linear span of the elements $T^n X^i$, $n \geq 0$, $1 \leq i \leq q$, coincides with M).

3. Non-unicellularity of the operators $S\Lambda$ in l^2 . Consider again the operators $T = S\Lambda$ in the space l^2 . We shall now show that conditions of the “regularity” type for the decrease of $|\lambda_k|$, $k \rightarrow \infty$ (see Theorem 1) are essential for the unicellularity of such operators. In this connection one may note that the very fact of the existence of non-unicellular operators $S\Lambda$ in the space l^2 such that $\lim_k \lambda_k = 0$, $\lambda_k \neq 0$, $k \geq 0$, is important. In this sense the question is settled by the following example:

Example. The operator $T = S\Lambda$ in the space l^2 such that $\lambda_n = 2^{-n}$.

if $n \neq 2^p + 1$, $p \geq 1$, and $\lambda_n = 2^{-(p+1)^2}$, if $n = 2^p + 1$, $p \geq 1$, is not unicellular.

However, a more general assertion also holds.

Theorem 3. Consider in the space l^2 the operator $T = S\Lambda$, $\lambda_k \neq 0$; $|\lambda_k| \geq |\lambda_{k+1}|$, $k = 0, 1, \dots$. Let $P = \{n_k\}_{k=1}^\infty$ be a sequence of natural numbers n_k , $k = 1, 2, \dots$, and $n_{k+1}/n_k > \kappa > 1$, $k \geq 1$. Then there exists a sequence $\{\mu_k\}_1^\infty$,

$\mu_k > 0$, $\mu_k \downarrow 0$, $k \rightarrow \infty$, such that, if $\tilde{T} = S\tilde{\Lambda}$, where: $\tilde{\lambda}_n = \lambda_n$, if $n \notin P$, and $\tilde{\lambda}_n = \mu_k$, if $n = n_k$, $k = 1, 2, \dots$, with $|\tilde{\lambda}_k| \leq \mu_k$, $k \geq 1$, then the operator $\tilde{T}$ is not unicellular.

4. Operators $S\Lambda$ in the space l_Z^2 . Let Z be the set of all integers and

$$l_Z^p = \{X = \{X_k\}_{k \in Z} : X_k \in C^1, k \in Z; \|X\| = (\sum_k |X_k|^p)^{1/p} < \infty\}.$$

Define in the space l_Z^p the shift operator S by the equalities $SX = Y$, $X \in l_Z^p$, where $X = \{X_k\}_{k \in Z}$, $Y = \{Y_k\}_{k \in Z}$ and $Y_k = X_{k-1}$, $k \in Z$, and the operator Λ of multiplication by the (bounded) sequence $\{\lambda_k\}_{k \in Z}$ by the equalities $\Lambda X = Y$, $X \in l_Z^p$, where $X = \{X_k\}_{k \in Z}$, $Y = \{Y_k\}_{k \in Z}$ and $Y_k = \lambda_k X_k$, $k \in Z$. If λ is a complex number ($\lambda \neq 0$), put $\text{sign } \lambda = \lambda/|\lambda|$, and let $m_k = \text{sign}(\lambda_0 \dots \lambda_{k-1})$, $k \geq 1$, $m_k = \text{sign}(\lambda_{-1} \dots \lambda_{k-1})^{-1}$, $k \leq 0$ (we assume, of course, that $\lambda_k \neq 0$, $k \in Z$). Finally, let M be the operator of multiplication by the sequence $\{m_k\}_{k \in Z}$, acting in the space l_Z^2 , and $P_{WX} = \{Y_k\}_{k \in Z}$, $Y_k = X_k$, $k \in W$, and $Y_k = 0$, $k \notin W$, where W is a subset of Z .

Theorem 4*. Consider the operator $T = S\Lambda$ in the space l_Z^2 and let $\lambda_0 = 1$, $\lambda_k \neq 0$, $k \in Z$. Then we have:

1) The following assertions are equivalent:

- a) T has nontrivial invariant subspaces;
 - b) Z is decomposed into a finite number of arithmetic progressions $Z_1, \dots, Z_n$, on each of which the modulus $|\lambda_p|$, $p \in Z_i$, $i = 1, \dots, n$, is constant;
 - c) there exists a natural number n such that $|T|^n = rS^n$, where $r > 0$ and $|T|^n = S^n|\Lambda|$, $|\Lambda|$ is the operator of multiplication by the sequence $\{|\lambda_k|\}_{k \in Z}$.
- 2) Let $Z_1, \dots, Z_n$ be disjoint arithmetic progressions with difference n and $\bigcup_1^n Z_i = Z$. If $|\lambda_k| = r_i$, $k \in Z_i$, where $r_i > 0$, $i = 1, \dots, n$, then the following assertions are equivalent:
- a) the subspace H , $H \subset l_Z^2$, reduces the operator T ;
 - b) $H = M^{-1}H_0$, where

$$H_0 = \sum_{i=1}^n \oplus P_{Z_i} H_0 \equiv \sum_{i=1}^n H_i$$

and $SH_i = H_{i+1}$, $1 \leq i \leq n-1$, $SH_n = H_1$, $H_1 = \{\{a_{nk}\}_{k \in Z} : \{a_p\}_{p \in Z} \in \widehat{H}\}$, where $\widehat{H}$ is a subspace of l_Z^2 , $S\widehat{H} = \widehat{H}$;

c) $H = M^{-1}(\chi_E L^2)$, $E \subset \{\lambda : |\lambda| = 1\}$, $e^{2\pi i/n} E = E$. (Here L^2 is the space of measurable functions on the circle $\{\lambda : |\lambda| = 1\}$, square-summable, and $\chi_E L^2 = \{\chi_E f : f \in L^2\}$, χ_E is the characteristic function of the set E . The correspondence of a function to the sequence of its Fourier coefficients is the

natural isomorphism of the spaces L^2 and l_Z^2 ; it is precisely in this sense that the equality in item c) is to be understood.)

The proof of Theorem 4 is based on the following lemma, useful also in some other questions.

Lemma. *Let T be a continuous operator in a Hilbert space $\mathcal{H}$ such that $T = VR$, where V is isometric and R is self-adjoint. Suppose, moreover, that the operator R is one-to-one (or, equivalently, $R\mathcal{H}$ is dense in $\mathcal{H}$). In order that a subspace*

* An analogous assertion can also be formulated for the operators $S\Lambda$ in the spaces $l^2\{H_i\}_0^\infty$, but the scope of the present note does not permit doing this (see (5) for the case $\lambda_k = 1$).

$L, L \subset \mathcal{H}$, reduces the operator T , it is necessary and sufficient that it reduce the operators V and R .

We now turn to the invariant (but not necessarily reducing) subspaces of the operators $S\Lambda$ in l_z^2 . It is easy to see that the subspaces

$$l_k^2 = \{X : X \in l_z^2, P_{iX} = 0, i < k\}, \quad k \in Z,$$

are invariant with respect to any operator of the form $T = S\Lambda$. The unicellularity of such an operator means that the subspaces l_k^2 , $k \in Z$, exhaust all T -invariant subspaces of l_z^2 . It is easy to give examples of non-unicellular operators $S\Lambda$ in l_z^2 (operators having reducing subspaces or having at least one eigenvalue).

From the results of the preceding section it also follows easily that the operator $T = S\Lambda$ is non-unicellular in the space l_z^p , if at least one of the operators $T_\pm = S\Lambda_\pm$ is non-unicellular in the space l^p , where $\Lambda_+(\Lambda_-)$ is the operator of multiplication by the sequence $\{\lambda_k\}_{k=0}^\infty$ ($\{\lambda_{-k}\}_{k=0}^\infty$, respectively).

However, the general question of the existence of unicellular operators of the form $T = S\Lambda$ in the space l_z^p remains open.

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Received
24 II 1966

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Note: Figure translations are in progress. See original paper for figures.

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