

THE GROUP OF MOTIONS OF A LINE

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Abstract

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MATHEMATICAL PHYSICS

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THE GROUP OF MOTIONS OF A LINE AND THE GEOMETRIC THEORY OF FIELDS

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1. As is known, gauge fields A_μ^a are introduced on the basis of the condition of covariance of equations of the Gelfand-Yaglom type ⁽¹⁾:

$$L^\mu \nabla_\mu \psi + \hat{\lambda} \psi = 0 \quad (1)$$

with respect to localized gauge transformations: $\delta\psi = \varepsilon^a(x) I_a \psi$ (I_a are the generators of a representation of some group).

The covariant derivative ∇_μ is defined by the commutation rule:

$$[\nabla_\mu \delta] = I_a \delta A_\mu^a,$$

where

$$\delta A_\mu^a = \varepsilon^b f_{bc}^a A_\mu^c + \partial_\mu \varepsilon^a. \quad (2)$$

The geometric theory of the potentials A_μ^a ^(2,3), according to which A_μ^a are regarded as coefficients of an affine connection of an internal space, presupposes a certain single-valued process of raising parametric indices, with the generators I_a and I^a related by the relation:

$$[I^a I_b] = f_b^{ac} I_c. \quad (3)$$

2. Relation (3) may be regarded as the structure formula of the given group only in the case when there exists a Casimir operator

$$H = g^{ab} I_a I_b, \quad \text{where } g^{ab} = f_m^{al} f_l^{bm}, \quad \text{and } [I^a I^b] = f_c^{ab} I^c.$$

From $[HI_a] = 0$ it follows ⁽⁴⁾ that

$$(g^{ab}f_{al}^c + g^{lc}f_{al}^b)I_c I_b = 0,$$

and hence $f_a^{bc} = g^{bl}f_{al}^c$.

But from $f_a^{bc} = g^{bl}f_{al}^c$, obviously, it follows that $f_a^{ac} = 0$. In other words, I_a and I^a form a basis of the algebra of a single structure under the condition that this structure is given by $f_{abc} = f_{[abc]}$, i.e., by constants antisymmetric in all indices. In papers ^(2,3) we essentially restricted ourselves to groups with structure $f_{[abc]}$, i.e., locally compact ones ⁽⁴⁾.

3. Let us admit consideration of an equation of a more general type

$$L^a X_a \psi + \hat{\lambda} \psi = 0, \quad (4)$$

where

$$X_a = \xi_a^i \frac{\partial}{\partial x^i}$$

transforms according to the regular representation of the group

$$\delta X_a = -\varepsilon^b f_{ba}^c X_c.$$

The covariance condition (4) has the form:

$$[L_b^a I] = f_{b\ c}^a L^c. \quad (5)$$

(5), obviously, passes into (3) for groups of structure $f_{[abc]}$.

4. One of the simplest examples of a group of a more general type ($f_{b\ c}^a$ are not antisymmetric in all indices) is the one-dimensional model of the Poincaré group of structure $[X_1 X_2] = -X_2$, isomorphic to the group of motions of the line I (5).
5. Let us investigate the representation I . We find the generators of the regular representation I in differential form. According to (5), introducing the regular representation on the set of left cosets with respect to the Abelian subgroup

$$u(t) = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix},$$

we obtain

$$X_\omega f = \left[\frac{df}{dt}(u\omega(t)) \right]_{t=0} = \frac{\partial f}{\partial \varepsilon^a} \varepsilon^{a'}(0).$$

Starting from the fundamental representation I

$$S(\varepsilon, \eta) = \begin{pmatrix} e^\eta & 0 \\ \frac{\varepsilon}{\eta} \operatorname{sh} \eta & e^{-\eta} \end{pmatrix},$$

where ε, η are parameters of I ; $\eta = \eta_1 + \eta_2$;

$$\frac{\varepsilon}{\eta} \operatorname{sh} \eta = e^{\eta_2} \frac{\operatorname{sh} \eta_1}{\eta_1} \varepsilon_1 + e^{-\eta_1} \frac{\operatorname{sh} \eta_2}{\eta_2} \varepsilon_2,$$

we obtain

$$X_1 = \frac{1}{2} \left[\frac{\partial}{\partial \eta} + \frac{\varepsilon}{\eta} \left(1 - \frac{e^{-\eta} \eta}{\operatorname{sh} \eta} \right) \frac{\partial}{\partial \varepsilon} \right]; \quad X_2 = \frac{e^{-\eta}}{\operatorname{sh} \eta} \frac{\partial}{\partial \varepsilon}. \quad (6)$$

Then from

$$\frac{\partial \psi_{mn}(\eta)}{\partial \eta} + \frac{m}{\eta} \left(1 - \frac{e^{-\eta} \eta}{\operatorname{sh} \eta} \right) \psi_{mn}(\eta) = n \psi_{mn}(\eta),$$

where $f_{mn}(\varepsilon, \eta) = \varepsilon^m \psi_{mn}(\eta)$, it follows that

$$S_{nn'}^{(p,k)}(\varepsilon, \eta) = \sum_{m=0}^p \sqrt{\binom{n}{m}} \varepsilon^m \eta^{-m} \operatorname{sh}^m \eta e^{(2n-m)\eta} \delta_{n-m, n'}; \quad (7)$$

$$0 \leq m \leq p; \quad k - p \leq n \leq k; \quad \psi_{mn}(\eta) \sim \operatorname{sh}^m \eta^{-m} e^{(2n-m)\eta}.$$

In this case

$$X_2 f_{mn}(\varepsilon, \eta) = \sqrt{mn} f_{m-1, n-1}(\varepsilon, \eta)$$

(X_2 shifts the columns $S_{nn'}^{(p,k)}$ to the right).

The dimension $s(p, k)$ is specified, as usual, by the character of I :

$$\chi(\eta) = e^{(2k-p)\eta} \frac{\operatorname{sh}(p+1)\eta}{\operatorname{sh} \eta}, \quad s(p, k) = \chi(0) = p + 1. \quad (8)$$

From this follows the addition formula for I :

$$T_g f_{mn}(\varepsilon_2 \eta_2) = \sum_{k=0}^m \sum_{p=0}^{m-k} g_{mn} \binom{m}{k} \binom{m-k}{p} e^{-2p\eta_1} \operatorname{ch}^k \eta f_{kn}(\varepsilon' \eta') f_{m-k, n-k}(\varepsilon, \eta).$$

The number k is not an independent weight, since

$$S_{nn'}^{(p,k)} = e^{2(k-p)} S_{nn'}^{(p,p)}.$$

The generators of the representation of weight p

$$M_1 e_n = n e_n, \quad M_2 e_n = \sqrt{n} e_{n-1} \quad (9)$$

follow from the expressions

$$X_1 S_{nn'}^{(p,k)}(\varepsilon, \eta) = n S_{nn'}^{(p,k)}(\varepsilon, \eta),$$

$$X_2 S_{n+1,n'}^{(p,k)}(\varepsilon, \eta) = \sqrt{nm} S_{n,n'}^{(p,k)}(\varepsilon, \eta).$$

It is easy to see that $f_{n,m}(\varepsilon, \eta)$ form a basis of the irreducible representation $S^{(p,k)}$.

6. In the particular case, from (7) we obtain the infinite-dimensional representation

$$S_{nn'}^{(\infty)} = \sum_{m=0}^{\infty} \binom{n}{m}^{1/2} \left(\frac{\varepsilon}{\eta} \operatorname{sh} \eta \right)^m e^{(2n-m)\eta} \delta_{n-m,n'}, \quad (10)$$

$$S_{nn'}^{(p,k)} \rightarrow S_{nn'}^{(\infty)} \quad (p \rightarrow \infty).$$

From (9) follows (3)

$$S_{nn'}^{(\infty)} = \exp(\eta M_1) \exp \left[\left(\frac{\varepsilon}{\eta} e^\eta \operatorname{sh} \eta \right) M_2 \right].$$

It turns out that relation (5) is realized only in the case of the infinite-dimensional representation (10).

Indeed, in case I

$$[L^1 M_1] = [L^1 M_2] = 0; \quad [M_1 L^2] = L^2,$$

$$[L^2 M_2] = L^1; \quad [M_1 M_2] = -M_2. \quad (11)$$

From Schur's lemma ^(5,6) it follows that $L^1 = \lambda E$, but then (11) reduces to Bose statistics:

$$[A^+A^-] = -E; \quad M_1 = A^+A; \quad M_2 = A^-; \quad L' = -E; \quad L^2 = A^+, \quad (12)$$

where $A^+ = \sqrt{n+1} \delta_{n+1,n'}$, $A^- = \sqrt{n} \delta_{n-1,n'}$ ($n, n' = 0, 1, 2, \dots$).

Then the covariance conditions (5) for the infinite system of coupled equations are written as

$$(\delta_{nn'} X_1 + \sqrt{n+1} \delta_{n+1,n'} X_2) \psi_{n'} + \lambda \psi_n = 0 \quad (n, n' = 0, 1, 2, \dots),$$

where $X_1 = \frac{1}{2}(x_1 \partial / \partial x_1 - x_2 \partial / \partial x_2)$, $X_2 = x_2 \partial / \partial x_1$ (x_1, x_2 are the variables of the space of the fundamental representation $I : S(1, \frac{1}{2})$); with respect to I it is realized only on the infinite-dimensional representation $S_a^{(\infty)}$ (12).

7. Let us investigate the invariants I : $\omega = \langle e_\mu | L^a | e_\nu \rangle$ ($a = 1, 2$). In the case $S(1, \frac{1}{2})$, the invariant I is given by the convolution $\omega_{\mu\nu} = \langle e_\mu | c | e_\nu \rangle$, where

$$c = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad e = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$$

belongs to the representation $S(1, \frac{1}{2})$ given by (7):

$$S(\varepsilon, \eta) = \begin{pmatrix} e & 0 \\ \frac{\varepsilon}{\eta} \operatorname{sh} \eta & e^{-\eta} \end{pmatrix}.$$

Then the invariant convolution is

$$\langle e_\mu | S^+ c S | e_\nu \rangle = \langle e_\mu | c | e_\nu \rangle.$$

In this sense $S(1, \frac{1}{2})$ are indeed analogous to spinors. In order that invariant convolutions also exist in the case of representations of arbitrary weight $S^{(p,k)}$, let us note that $S^{(p,k)}$ is contained among the irreducible representations into which the direct product of p spinors decomposes:

$$S(1, \frac{1}{2}) \times \dots \times S(1, \frac{1}{2}).$$

Thus, in case I the device called the (6) folded direct product is also valid.

It can be shown that $S^{(p+p', k+k')}$ is always contained in the decomposition $S^{(p,k)} \times S^{(p',k')}$, and hence, in the general case, the invariant of $S^{(p,k)}$ has the form:

$$\omega_{\mu\nu} = \left\langle e_{\mu} \left| \underbrace{c \times \cdots \times c}_p \right| e_{\nu} \right\rangle.$$

8. Let us carry out the decomposition for the product of two conjugate spinors. Obviously, in the general case $S^{(p,k)}$ the formulas obtained extend by complete induction:

$$US(1, 1/2) \times [S^{-1}(1, 1/2)]^t U^{-1} = U \begin{pmatrix} e^{\eta} & 0 \\ \frac{\varepsilon}{\eta} \operatorname{sh} \eta & e^{-\eta} \end{pmatrix} \times \begin{pmatrix} e^{-\eta} & -\frac{\varepsilon}{\eta} \operatorname{sh} \eta \\ 0 & e^{\eta} \end{pmatrix} U^{-1} \\ = \begin{vmatrix} e^{2\eta} & 0 & 0 & 0 \\ -\sqrt{2} \frac{\varepsilon}{\eta} e^{\eta} \operatorname{sh} \eta & 1 & 0 & 0 \\ -\varepsilon^2 \frac{\operatorname{sh}^2 \eta}{\eta^2} & \sqrt{2} e^{-\eta} \frac{\varepsilon}{\eta} \operatorname{sh} \eta & e^{-2\eta} & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = \begin{bmatrix} S^{(2,1)} & 0 \\ 0 & f_{00} \end{bmatrix}.$$

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CITED LITERATURE

- ¹ I. M. Gel' fand, A. M. Yaglom, ZhETF, 18, 703 (1948).
- ² G. A. Sokolik, N. P. Konopleva, Nuclear Phys., 72, 667 (1965); G. A. Sokolik, *Group Methods in the Theory of Elementary Particles*, 1965.
- ³ G. A. Sokolik, N. P. Konopleva, DAN, 154, 310 (1964).
- ⁴ M. Hamermesh, *Group Theory and its Applications to Physical Problems*, 1962.
- ⁵ N. Ya. Vilenkin, *Special Functions and the Theory of Group Representations*, "Nauka," 1965, pp. 146-152.
- ⁶ F. Murnaghan, *The Theory of Group Representations*, IL, 1950.
- ⁷ R. Utiyama, Phys. Rev., 101, 1597 (1956).
- ⁸ G. A. Sokolik, N. P. Konopleva, DAN, 169, No. 3 (1966).

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