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Abstract

Full Text

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THEORY OF ELASTICITY

F. P. KOCHANOV

SOLUTION OF THE GENERALIZED PROBLEM OF I. Ya. SHTAERMAN

(Presented by Academician A. Yu. Ishlinskii, May 7, 1966)

The problem of the contact of elastic cylinders of nearly equal radii in the absence of friction was first posed and approximately solved by I. Ya. Shtaerman⁽¹⁾. Subsequently, under various assumptions, it was considered many times (see the bibliography in⁽²⁾). However, the solutions found, except for⁽¹⁾, are of a general theoretical character.

Below we shall solve the problem of I. Ya. Shtaerman with friction taken into account, which is important for engineering practice. For the solution we shall use the Kolosov–Muskhelishvili potentials $\varphi(z)$, $\psi(z)$, taking them in the form obtained in⁽³⁾ for an arbitrary load. In order to simplify the exposition, let us assume that the inner cylinder is loaded by a vertical force P_0 and by a couple with moment M_0 , applied at its center (Fig. 1), where

$$P_0 = R_i \int_l (p_i \cos \varphi + \tau_i \sin \varphi) d\varphi,$$

$$M_0 = R_i^2 \int_l \tau_i d\varphi. \quad (1)$$

Fig. 1. Scheme of contact of cylinders

From the prescribed external load and from the contact forces $p_i(\varphi)$, $\tau_i(\varphi)$, we find the components $u_1(\varphi)$, $v_1(\varphi)$ and $u_2(\varphi)$, $v_2(\varphi)$ of the displacements of points on the boundaries of the cylinders (index 1 refers to the outer cylinder, and 2 to the inner one).

Substituting the components $u_i(\varphi)$ and $v_i(\varphi)$ into the expression for the curvature of a plane deformed curve⁽³⁾, we find the curvatures K_1 and K_2 of the

boundaries of the cylinders and require that they coincide identically in the contact zone l . As a result we obtain the equation

$$\frac{B}{8\pi} \int_l (\tau - p') \operatorname{ctg} \frac{\varphi - \alpha}{2} d\varphi - \frac{1 + \chi_1}{8\pi\mu_1} R_2 \int_l p d\varphi - A \frac{p + \tau'}{4} - \frac{1}{2\pi} \left(\frac{\chi_1}{\mu_1} + \frac{1}{\mu_2} \right) P_0 \cos \alpha = \delta, \quad \alpha_1 \leq \alpha \leq \alpha_2. \quad (2)$$

Here it has been taken into account that $\tau_1 = \tau_2 = \tau$, $p_1 = p_2 = p$, and the notation has been introduced: α_1, α_2 are the limiting angles of the contact zone, $\delta = R_1 - R_2$; R_1, R_2 are the radii of the cylinders.

An analogous equation was obtained by another method in ⁽²⁾; however, its solution is not given there and, with the general indications made by the author, cannot be obtained in the adopted formulation.

In the general case ⁽⁴⁾, in the contact zone, along with regions of slip, where we shall assume Coulomb's law to be valid,

$$\tau_i = f p_i \quad (i = 1, 2), \quad (3)$$

there may be, in advance, an unknown segment l_0 of rigid adhesion. The equation determining l_0 is found on the basis of the assumption that on this segment the lengths of the deformed boundaries are identical. This gives

$$\int_{l_0} (R_1 + v'_1 \cos \varphi - u'_1 \sin \varphi) d\varphi = \int_{l_0} (R_2 + v'_2 \cos \varphi - u'_2 \sin \varphi) d\varphi. \quad (4)$$

Since relation (4) is valid for any arc whose length is less than l_0 , the integrands in it are identical. Taking into account the values of the components $u_i(\varphi)$, $v_i(\varphi)$, we find

$$\frac{B^0}{8\pi} \int_l \tau(\varphi) \operatorname{ctg} \frac{\varphi - \alpha}{2} d\varphi - \frac{P_0}{4\pi} \left(\frac{\kappa_1}{\mu_1} - \frac{1}{\mu_2} \right) \cos \alpha - \frac{A^0}{4} p(\alpha) - \frac{1 + \kappa_1}{8\pi\mu_1} R_1 \int_l p d\varphi = \delta, \quad \alpha_1^0 \leq \alpha \leq \alpha_2^0, \quad (5)$$

where α_1^0, α_2^0 are the end angles of the segment of rigid adhesion.

Equations (2), (5) can be reduced to Prandtl equations, and then to a system of Fredholm equations. However, there is no point in doing this, since from the latter system it is difficult to obtain an effective solution. Therefore, for engineering purposes it is advisable to solve these equations by one of the numerical methods, for example by the method of finite differences. The advantages of this method are simplicity, uniformity of the computations, and the use of standard programs in solving on computers.

To reduce relations (2), (5) to Fredholm integral equations of the first kind with an absolutely integrable kernel, consider the functions

$$\Phi(\alpha) = \frac{B}{4\pi} \int_l \left(2\tau \sin \frac{\alpha - \varphi}{2} - p \cos \frac{\alpha - \varphi}{2} \right) \ln \operatorname{tg} \left| \frac{\alpha - \varphi}{4} \right| d\varphi + \frac{1}{4} \int_l p \sin \left| \frac{\alpha - \varphi}{2} \right| d\varphi + \frac{A}{8} \left(\int_{\alpha_1^0}^{\alpha} \tau \cos \frac{\alpha - \varphi}{2} d\varphi - \right.$$

$$\Phi^0(\alpha) = \frac{B^0}{4\pi} \int_l 2\tau \sin \frac{\alpha - \varphi}{2} \ln \operatorname{tg} \left| \frac{\alpha - \varphi}{4} \right| d\varphi + \frac{A^0}{4} \int_l p \sin \left| \frac{\alpha - \varphi}{2} \right| d\varphi.$$

Differentiating these functions twice with respect to α and solving the resulting equations, we find, taking (2), (5) into account,

$$\frac{A}{4} \int_l p \left(\sin \left| \frac{\alpha - \varphi}{2} \right| - \cos \frac{\alpha}{2} \sin \left| \frac{\varphi}{2} \right| \right) d\varphi + \frac{A}{4} \left(\int_{\alpha_1^0}^{\alpha} - \int_{\alpha}^{\alpha_2^0} \right) \tau \cos \frac{\alpha - \varphi}{2} d\varphi - \frac{2P_0}{3\pi} \left(\frac{\mu_1}{\mu_2} + \frac{1}{\mu_2} \right) \left(\cos \alpha - \cos \frac{\alpha}{2} \right) +$$

$$\frac{B^0}{2\pi} \int_l \tau \left(\sin \frac{\alpha - \varphi}{2} \ln \operatorname{tg} \left| \frac{\alpha - \varphi}{4} \right| + \cos \frac{\alpha}{2} \sin \frac{\varphi}{2} \ln \operatorname{tg} \left| \frac{\varphi}{4} \right| \right) d\varphi + \frac{A^0}{4} \int_l p \left(\sin \left| \frac{\alpha - \varphi}{2} \right| - \cos \frac{\alpha}{2} \sin \left| \frac{\varphi}{2} \right| \right) d\varphi - \frac{P_0}{3\pi} \left(\right.$$

Equations (6), (7), together with (1), (3), completely solve the generalized problem of I. Ya. Shtaerman (with allowance for friction). By the finite-difference method, the problem reduces to the solution of a system of nonlinear algebraic equations

$$\begin{aligned} D_1 \sum_1^n & \left[q_k (\Psi_{s+k} + \Psi_{s-k+1} - 2\Psi_k \cos \frac{s}{2}\beta) \right. \\ & \left. - 2q_{n+k} (\Phi_{s+k} - \Phi_{s-k+1} - 2\Phi_k \cos \frac{s}{2}\beta) \right] \\ & - \frac{8}{3} D_2 \sin \frac{\beta}{2} \sin \frac{s}{2}\beta \sin \frac{3s}{4}\beta \sum_1^n \left(q_k \cos \frac{2k-1}{2}\beta + q_{n+k} \sin \frac{2k-1}{2}\beta \right) \quad (8) \\ & - D_3 \bar{\beta} \left(1 - \cos \frac{s}{2}\beta \right) \sum_1^n q_k - 2D_4 \sin \frac{\beta}{4} \sum_1^s \left(2q_k \sin \frac{2s-2k+1}{2}\beta \right. \\ & \left. - q_{n+k} \cos \frac{2s-2k+1}{4}\beta \right) = 2 \left(1 - \cos \frac{s}{2}\beta \right). \end{aligned}$$

where $\beta = \varphi_0/n$; $\bar{\beta}$ is the angle β in radians; φ_0 is the angle of the boundary of the contact zone; $s = 1, 2, \dots, n$; $n = \varphi_0/\beta$;

$$\begin{aligned}
& D_1^0 \sum_1^n q_{n+k} \left(\Phi_{s+k} - \Phi_{s-k+1} - 2\Phi_k \cos \frac{s}{2}\beta \right) + \\
& + \frac{8}{3} D_2^0 \sin \frac{\beta}{2} \sin \frac{s}{4}\beta \sin \frac{3s}{4}\beta \sum_1^n \left(q_k \cos \frac{2k-1}{2}\beta + q_{n+k} \sin \frac{2k-1}{2}\beta \right) + \\
& + D_3^0 \bar{\beta} \left(1 - \cos \frac{s}{2}\beta \right) \sum_1^n q_k + 2D_4^0 \sin \frac{\beta}{4} \sum_1^s q_k \sin \frac{2s-2k+1}{4}\beta = \\
& = - \left(1 - \cos \frac{s}{2}\beta \right),
\end{aligned} \tag{9}$$

where $s = 1, 2, \dots, t$; $t = \psi_0/\beta$; ψ_0 is the angle of the boundary of the adhesion segment. To these are added the relations given by law (3), $q_{n+m} = f q_m$, $m = t+1, t+2, \dots, n$, where f is the coefficient of sliding friction, and the matching conditions of the slip segments with the adhesion segment $q_{n+t} = q_{n+t+1}$.

The following notation has been introduced above:

$$\begin{aligned}
q_k &= p_k R_2 \frac{\varkappa_2 + 1}{4\pi\delta\mu_2}, & q_{n+k} &= \tau_k R_2 \frac{\varkappa_2 + 1}{4\pi\delta\mu_2}, \\
\left. \begin{matrix} D_1 \\ D_1^0 \end{matrix} \right\} &= 1 + b + \frac{c}{bc}, & \left. \begin{matrix} D_2 \\ D_2^0 \end{matrix} \right\} &= \frac{4}{1} b \left(1 - \frac{\mu_2 \mp \mu_1}{\mu_1(\varkappa_1 + 1)} \right), \\
\left. \begin{matrix} D_3 \\ D_3^0 \end{matrix} \right\} &= \frac{2}{1} b \left(1 + \frac{0}{c} \right), & \frac{D_4}{D_4^0} &= \frac{\pi}{d_2} \frac{e_2}{e_1} \left(\frac{e_2 - e_1}{e_2} \pm c \right), & b &= \frac{d_1}{d_2}, \\
d_i &= (\varkappa_i + 1)/\mu_i, & e_i &= (\varkappa_i - 1)/\mu_i, & \varkappa_i &= 3 - 4\nu_i, \\
\mu_i &= E_i/2(1 + \nu_i),
\end{aligned}$$

ν_i is Poisson' s ratio, E_i is Young' s modulus,

$$\begin{aligned}
\Psi_r &= \sin \frac{r}{2}\beta \ln \operatorname{tg} \frac{r}{4}\beta - \sin \frac{r-1}{2}\beta \ln \operatorname{tg} \left| \frac{r-1}{4} \right| \beta, \\
\Phi_r &= \left(\cos \frac{r}{2}\beta \ln \operatorname{tg} \frac{r}{4}\beta - \ln \sin \frac{r}{2}\beta \right) - \\
&\quad - \left(\cos \frac{r-1}{2}\beta \ln \operatorname{tg} \left| \frac{r-1}{4} \right| \beta - \ln \sin \left| \frac{r-1}{2} \right| \beta \right).
\end{aligned}$$

Equations (8), (9) have been derived under the assumption that $M_0 = 0$.

Fig. 2. Dependence of the contact zone on the parameters according to Hertz (1), according to the author (2 $-c = 0.010$; 3 $-c = 0.100$), according to Shtaerman (4), and according to Panasyuk (5)

Figure 2: Fig. 2. Dependence of the contact zone on the parameters according to Hertz (1), according to the author (2 $-c = 0.010$; 3 $-c = 0.100$), according to Shtaerman (4), and according to Panasyuk (5)

The method of solution presented here may be called inverse, since it reduces to the following: we specify the properties of the materials, the quantities l

and l_0 ; we divide the zone l into n parts, assuming that an integer number of such parts fits on the segment l_0 and that the functions $p(a)$ and $\tau(a)$ are constant on each part. After solving the systems we find p and τ on each segment, the magnitude of the external load for which the assumed zone l is possible, and, finally, the coefficient f for which the assumed segment l_0 is possible.

As the simplest case, let us consider the case when $l_0 = 0$ and $f = 0$. Taking the materials to be identical (with $\nu = 0.3$), we arrive at the problem solved by I. Ya. Shtaerman, which now reduces to solving equations (8).

Fig. 2. Dependence of the contact zone on the parameters according to Hertz (1), according to the author (2 $-c = 0.010$; 3 $-c = 0.100$), according to Shtaerman (4), and according to Panasyuk (5)

The system (8) will be close to the system considered by I. Ya. Shtaerman⁵ (Appendix (2), equations (19)), but differs from it, first, because of the difference in the character of the application of the external load and, second, because in it the coefficients D depend on $c = \delta/R_2$, whereas in work⁵, because of a misprint in formula (227), Ch. II, this dependence was not taken into account. As a result, the quantity l found in work⁵ depends only on P_0 , E , and δ , while from system (8) there also follows its dependence on δ/R_2 (Fig. 2).

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