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Abstract

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MATHEMATICS

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COMPLETENESS OF TRANSLATES OF A FUNCTION IN CERTAIN SPACES

(Presented by Academician V. I. Smirnov on 28 V 1966)

1°. The completeness* of the system of functions $\{f_x(\tau)\}$, $f_x(\tau) = f(x + \tau)$, $-\infty < x < \infty$, in the spaces L_α ,

$$\|f\| = \int_{-\infty}^{\infty} |f(\tau)| e^{\alpha(\tau)} d\tau,$$

where $\alpha(\tau) = \alpha(-\tau) > 0$, $\alpha(\tau)$ is nondecreasing, $\alpha(\tau)/\tau$ is nonincreasing, $\alpha(\tau_1 + \tau_2) \leq \alpha(\tau_1) + \alpha(\tau_2)$, was studied in ⁽¹⁻³⁾. It turned out that the conditions for completeness depend on the rate of growth of $\alpha(\tau)$. Shilov ⁽²⁾ showed that, for $\alpha(\tau) = o(\ln \tau)$, the system $\{f_x(\tau)\}$ is complete in L_α if and only if

$$\tilde{f}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\tau) e^{ix\tau} d\tau \neq 0, \quad -\infty < x < \infty. \quad (1)$$

Korenblum ⁽³⁾ studied the case $\alpha(\tau) = c\tau$. He found that, instead of (1), the necessary and sufficient conditions** are

$$\tilde{f}(z) \neq 0, \quad |\operatorname{Im} z| \leq c, \quad -\infty < \operatorname{Re} z < \infty,$$

$$\overline{\lim}_{x \rightarrow \infty} \ln |\tilde{f}(x)| / e^{\pi x/2c} = 0, \quad \overline{\lim}_{x \rightarrow -\infty} \ln |\tilde{f}(x)| / e^{-\pi x/2c} = 0. \quad (2)$$

The question naturally arises: for which $\alpha(\tau)$ is condition (1) sufficient for the completeness of the system $\{f_x(\tau)\}$ in L_α . Let N be the class of such functions $\alpha(\tau)$. In the present paper a sufficient condition is obtained for $\alpha(\tau)$ to belong to the class N , which is necessary if $\alpha(\tau) = o(\tau)$ and $\tau\alpha'(\tau)/\alpha(\tau)$ is nonincreasing. It is shown that this condition is at the same time, in an analogous way, necessary and sufficient for the regularity of L_α , regarded as a normed ring

with multiplication given by convolution. For nonregular rings L , necessary completeness conditions are obtained which are analogous to conditions (2).

2°. Theorem 1. *If*

$$\int_0^\infty \frac{\alpha(\tau) d\tau}{1 + \tau^2} < \infty, \quad (3)$$

then the ring L_α is regular and the system $\{f_x(\tau)\}$ is complete in L_α if and only if (1) holds.

Proof. We first show the regularity of L_α . From the description of the maximal ideals of L_α obtained in ⁽⁴⁾ it follows that it suffices, for arbitrary x_0 and $\varepsilon > 0$, to construct a function $\varphi(x) \in L_\alpha$ such that $\tilde{\varphi}(x_0) \neq 0$ and $\tilde{\varphi}(x) = 0$ for $|x - x_0| \geq \varepsilon$. We may assume that $x_0 = 0$ and

$$\alpha(x) \leq \int_1^x \frac{\alpha(\tau) d\tau}{\tau}$$

* A system $\{x_\nu\}$ of elements of a Banach space X is called **complete** in X if the closure of the linear span of the system coincides with X .

** In what follows, $\tilde{f}(x)$ denotes the Fourier transform of the function $f(x)$, defined by the equality in formula (1).

for $x \geq 2$. Put

$$\beta(\tau) = \alpha(\tau) \left(\int_\tau^\infty \frac{\alpha(\theta) d\theta}{1 + \theta^2} \right)^{-1/2}, \quad \tau \geq 0, \quad \beta(-\tau) = \beta(\tau).$$

Using the condition imposed on $\alpha(\tau)$, we obtain, for $A > 1$,

$$\int_0^\infty \frac{\beta(\tau) d\tau}{1 + \tau^2} < \infty; \quad \beta(A\tau) \leq A^2 \beta(\tau); \quad \lim_{x \rightarrow \infty} \left(\int_1^x \frac{\alpha(\tau) d\tau}{\tau} - \int_1^{x/A} \frac{\beta(\tau) d\tau}{\tau} \right) = -\infty. \quad (4)$$

Next, let $\omega(\tau)$ be the function inverse to $\beta(\tau)$, and for $x \geq 1$

$$K(x) = \exp \left(\int_1^x \frac{\alpha(\tau) d\tau}{\tau} \right), \quad N(x) = \exp \left(\int_0^{\omega(x)} \beta'(\tau) \ln \tau d\tau \right), \quad F(x) = \min_{\tau \geq 1} N(\tau) x^{-\tau}.$$

Then $N'(\beta(x))/N(\beta(x)) = \ln x$, and therefore $F(x) = N(\beta(x))x^{-\beta(x)}$, i.e.

$$F(x) = \exp \left(- \int_1^x \frac{\beta(\tau) d\tau}{\tau} \right). \quad (5)$$

Moreover, for $x \geq 1$ we have

$$\frac{N'(x)}{N(x)} = \ln \omega(x), \quad \ln \frac{N(x+1)}{N(x)} \geq \ln \omega(x), \quad \frac{N(x)}{N(x+1)} \leq \int_{x-1}^x \frac{d\theta}{\omega(\theta)},$$

hence, by virtue of (4),

$$\sum_{n=2}^{\infty} \frac{N(n)}{N(n+1)} \leq \int_0^{\infty} \frac{d\theta}{\omega(\theta)} = O(1) + \int_0^{\infty} \frac{\beta(\tau) d\tau}{1+\tau^2} < \infty. \quad (6)$$

Now put

$$\mu_2 = \mu_3 = 1/N(2), \quad \mu_j = N(j-3)/N(j-2),$$

$$4 \leq j < \infty \quad \text{and} \quad \psi(x) = \prod_{n=2}^{\infty} \frac{\sin \mu_n x}{\mu_n x}, \quad \psi_{\delta}(x) = \psi(\delta x).$$

By (6), $\psi(x)$ is an entire function of some finite degree. Since

$$|\psi(x)| = O(1) \min_{j \geq 2} N(j-1)|x|^{-j},$$

we have $|\psi(x)| = O(F(|x|))$. Hence, from (4), (5) it follows that

$$\int_{-\infty}^{\infty} |\psi_{\delta}(x)| e^{\alpha(x)} dx = O(1) \int_1^{\infty} F(\delta x) \exp \left(\int_1^x \frac{\alpha(\tau) d\tau}{\tau} \right) dx < \infty.$$

Thus $\psi_{\delta}(x) \in L_{\alpha}$ for $\delta > 0$, and from the Wiener-Paley theorem we obtain that $\tilde{\psi}_{\delta}(x) = 0$ for $|x| \geq \delta\sigma$, while $\tilde{\psi}_{\delta}(0) = 0$. Therefore the function $\varphi(x) = \psi_{\varepsilon/\sigma}(x)$ is the desired one, and the regularity of L_{α} is proved.

From the regularity of L_{α} and the results of G. E. Shilov ⁽²⁾ it follows that, in order to prove the second assertion, it is enough to show the density in L_{α} of functions with finite Fourier transforms. Let $p(x) \in L_{\alpha}$. Put

$$p_A(x) = Aq \int_{-\infty}^{\infty} p(x-t)\psi_A(t) dt, \quad q = \left(\int_{-\infty}^{\infty} \psi(t) dt \right)^{-1}.$$

Obviously, $p_A(x) \in L_{\alpha}$ and $\tilde{p}_A(x)$ is finite. We shall show that

$$\lim_{A \rightarrow \infty} \|p - p_A\| = 0.$$

For $\mu > 0$ and $A \rightarrow \infty$ we have

$$\begin{aligned} \int_{-\infty}^{\infty} \int_{|t| > \mu} |p(x) - p(x-t)| |\psi_A(t)| Aq dt e^{\alpha(x)} dx &= O(1) \|p\| \int_{|t| > A\mu} |\psi(t)| e^{\alpha(t)} dt, \\ \int_{-\infty}^{\infty} \int_{|t| < \mu} |p(x) - p(x-t)| |\psi_A(t)| Aq dt e^{\alpha(x)} dx &\leq \\ &\leq \int_{|t| < \mu} |\psi_A(t)| Aq \int_{-\infty}^{\infty} |p(x)e^{\alpha(x)} - p(x-t)e^{\alpha(x-t)}| dx dt + \\ &+ \max_{|x| < \infty, |t| < \mu} |e^{\alpha(x) - \alpha(x-t)} - 1| \int_{|t| < \mu} |\psi_A(t)| Aq dt = o(1) + O(\mu). \end{aligned}$$

Consequently,

$$\overline{\lim}_{A \rightarrow \infty} \|p - p_A\| = O(\mu).$$

Letting $\mu \rightarrow 0$, we obtain

$$\lim_{A \rightarrow \infty} \|p - p_A\| = 0,$$

as was required.

3°. **Theorem 2.** Let $\alpha(\tau) = o(\tau)$, let $\tau\alpha'(\tau)/\alpha(\tau)$ be nondecreasing,

$$\int_0^{\infty} \frac{\alpha(\tau) d\tau}{1 + \tau^2} = \infty.$$

Then the Fourier transforms of functions from L_{α} form a quasi-analytic class of functions, and the ring L_{α} is nonregular.

Theorem 3. If $\alpha(\tau)$ satisfies the conditions of Theorem 2 and the system $\{f_x(\tau)\}$ is complete in L_{α} , then (1) holds and, for real A ,

$$\overline{\lim}_{x \rightarrow \infty} \frac{\ln |\tilde{f}(x)|}{\xi(\frac{1}{2}\pi(x-A))} = 0, \quad \overline{\lim}_{x \rightarrow -\infty} \frac{\ln |\tilde{f}(x)|}{\xi(\frac{1}{2}\pi(x+A))} = 0, \quad (7)$$

where $\xi(x)$ is the inverse function of the function

$$\eta(x) = \int_0^x \frac{\alpha(\tau) d\tau}{1 + \tau^2}.$$

Let us note that (7) coincides with (2) when $\alpha(x) = cx$. Denote by $I_{\gamma,A}$ the set of functions $f \in L_\alpha$ for which

$$\overline{\lim}_{x \rightarrow \infty} \frac{\ln |\hat{f}(x)|}{\xi(\frac{1}{2}\pi(x-A))} \leq \gamma < 0.$$

Theorem 3 follows from the following assertion, which is of independent interest.

Theorem 4. *Let $\alpha(\tau)$ satisfy the conditions of Theorem 2. Then for arbitrary γ and A the set $I_{\gamma,A}$ contains a function whose Fourier transform does not vanish, and there exists a function $g(t)$, for which*

$$0 < \text{vrai sup}_{-\infty < \tau < \infty} |g(\tau)e^{-\alpha(\tau)}| < \infty, \quad \int_{-\infty}^{\infty} g(x-t)f(t) dt \equiv 0 \quad (8)$$

for all $f \in I_{\gamma,A}$.

We shall need the following lemmas.

Lemma 1. *Let $\omega(t)$ be the inverse function for the function $2\alpha(t)/\pi$,*

$$z = x + iy, \quad \Phi(z) = \prod_{n=1}^{\infty} \left(1 + \frac{z}{\omega(n)}\right) e^{-z/\omega(n)}.$$

Then we have

$$\frac{c_1}{1 + |y|^{c_2}} \leq \Phi(iy)e^{-\alpha(y)} \leq \Phi_1(y), \quad \Phi_1(y) = \Phi_1(-y);$$

$$\Phi_1(y) \uparrow, \quad \int_0^{\infty} \frac{\ln \Phi_1(y)}{1 + y^2} dy < \infty, \quad (9)$$

and, for $x \geq 0$,

$$|\Phi(x)| \leq c_4 e^{c_3 x} \exp \left(-\frac{2x}{\pi} \int_0^x \frac{\alpha(\tau) d\tau}{1 + \tau^2} \right), \quad (10)$$

$$|\Phi(z)| \geq \frac{c_5}{1 + |y|^{c_6}} e^{-c_7 x} \exp \left(-\frac{2x}{\pi} \int_0^x \frac{\alpha(\tau) d\tau}{1 + \tau^2} \right), \quad (11)$$

where c_i are positive constants.

* $1/g(t)$ depends on γ and A .

Lemma 2. If $f \in I_{\gamma,A}$, then for $x \geq 0$

$$\int_0^\infty |\tilde{f}(p)| e^{xp} dp \leq c_9 \exp \left(\frac{2x}{\pi} \int_0^x \frac{\alpha(\tau) d\tau}{1 + \tau^2} + c_8 x \right),$$

where c_8 depends only on γ, A .

Proof of Theorem 3. Put

$$R(y) = \left(\frac{\sin y}{y} \right)^B \frac{1}{\Phi(iy)}.$$

By virtue of 9, for sufficiently large B , $R(y) \in L_\alpha$, and from (11) it follows that, for $x > 0$, $\sigma > 0$,

$$|\tilde{R}(x)| \leq \exp \left(-\sigma x + c_{10}\sigma + \frac{2\sigma}{\pi} \int_0^\sigma \frac{\alpha(\tau) d\tau}{1 + \tau^2} \right).$$

Taking $\sigma = \xi(1/2\pi(x - H))$, for large H we obtain

$$|\overline{R}(x)| \leq \exp \left(-\frac{1}{2} H \xi(1/2\pi(x - H)) \right).$$

It is not difficult to show that, for any $\delta > 0$,

$$\lim_{x \rightarrow \infty} \frac{\xi(x - \delta)}{\xi(x)} = 0.$$

Therefore one can choose η so that

$$\tilde{H}(y) = e^{i\eta y} R(y) \in I_{\gamma,A},$$

and $\tilde{H}(y)$ does not vanish. The first assertion of the theorem is proved.

Next, put

$$\mu(z) = \frac{2}{\pi} \int_{-\infty}^\infty \frac{\ln \Phi_1(\rho) d\rho}{i\rho - (z + 1)}.$$

Then $\mu(z)$ is holomorphic for $\operatorname{Re} z > -1$, and by virtue of (9)

$$\operatorname{Re} \mu(z) \leqslant -\frac{z \ln \Phi_1(z)}{\pi} \int_{\rho > |z|} \frac{(x+1) d\rho}{(\rho-y)^2 + (x+1)^2}.$$

The function $s(z) = e^{\mu(z)}$ is holomorphic and bounded for $\operatorname{Re} z \geqslant 0$, and

$$|s(iy)| \leqslant \frac{1}{\Phi_1(|y|)}.$$

Put

$$f \in I_{\gamma, A} \quad \text{and} \quad f_1(x) = \int_{-\infty}^{\infty} e^{-(x-\tau)^2} f(\tau) d\tau.$$

In this case

$$\max_{-\infty < x < \infty} |f_1(x) e^{\alpha(x)}| < \infty, \quad f_1(x) \in I_{\gamma, A},$$

and, by Lemma 2, for $x \geqslant 0$,

$$|f_1(iz)| = O(1) \left| \int_{-\infty}^{\infty} \tilde{f}_1(p) e^{zp} dp \right| = O(1) \exp \left(\frac{2x}{\pi} \int_0^x \frac{\alpha(\tau) d\tau}{1+\tau^2} + c_8 x \right).$$

Hence, and from (9), it follows that for any real β the function

$$Q_\beta(z) = \Phi(z) s(z) e^{-2c_8 z} f_1(iz + \beta)$$

is holomorphic in the right half-plane, bounded on the axis y and on the positive semiaxis x , continuous for $x \geqslant 0$, and of order $\rho < 2$. Applying the Phragmén-Lindelöf principle, we find that $Q_\beta(z)$ is bounded in the right half-plane. Then, by Cauchy's theorem,

$$\int_{-\infty}^{\infty} \frac{Q_\beta(iy) dy}{(1+iy)^2} = 0.$$

Consequently, for $-\infty < \beta < \infty$,

$$\int_{-\infty}^{\infty} \Phi(iy) s(iy) e^{-2c_8 iy} f_1(\beta - y) \frac{dy}{(1-iy)^2} = \int_{-\infty}^{\infty} e^{-(\beta-\tau)^2} \int_{-\infty}^{\infty} f(\tau-v) g(v) dv d\tau = 0, \quad (12)$$

where

$$g(v) = \Phi(iv)s(iv)e^{-2icsv}(1+iv)^{-2}.$$

From (12), by the usual device, we derive

$$\int_{-\infty}^{\infty} f(\tau-v)g(v) dv = 0.$$

Since g does not depend on f and, by virtue of (10),

$$0 < \max_{-\infty < v < \infty} |g(v)e^{-\alpha(v)}| < \infty,$$

the last equality is equivalent to the last assertion of Theorem 3.

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Note: Figure translations are in progress. See original paper for figures.

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