

AN EXTREMALLY DISCONNECTED BICOMPACTUM OF WEIGHT $\aleph_c$ IS NONHOMOGENEOUS

1967

SovietRxiv

View the original and related papers at <https://sovietsrxiv.org/items/ru-196701.23422>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

UDC 513.83

MATHEMATICS

A. V. ARKHANGELSKII

AN EXTREMALLY DISCONNECTED BICOMPACTUM OF WEIGHT c IS NONHOMOGENEOUS*

(Presented by Academician P. S. Aleksandrov on 14 X 1966)

Recall that a bicom pactum is called extremally disconnected if the closure of each of its open subsets is open (see ⁽¹⁾). Below further information is given on the structure of extremally disconnected bicom pacta. The main results are Theorems 1 and 2.

Proposition 1. *Every infinite bicom pact subspace of an extremally disconnected space contains a subspace homeomorphic to the Čech extension of the natural numbers (denoted below by βN).*

Proof. Every infinite bicom pactum contains a countable discrete subspace. Therefore it suffices to prove Lemma 1.

Lemma 1. *The closure $[A]$ of an infinite countable discrete subspace A , contained in a bicom pact subspace of an extremally disconnected space X , is homeomorphic to βN .*

Obviously, $[A]$ is a bicom pact extension of the space A . Let $A_1 \subset A$, $A_2 \subset A$, and $A_1 \cap A_2 = \Lambda$. In view of the countability of the sets A_1 and A_2 and the discreteness of A , the sets A_1 and A_2 have in the space X disjoint open neighborhoods $\mathcal{U}_1$ and $\mathcal{U}_2$. We have $[\mathcal{U}_2] \cap \mathcal{U}_1 = \Lambda$, moreover $[\mathcal{U}_2] \supseteq A_2$ and $\mathcal{U}_1 \supseteq A_2$. By virtue of the extremal disconnectedness of the space X , the set $[\mathcal{U}_2]$ is open. Hence $[A_1] \cap [A_2] = \Lambda$, whence it follows that $[A]$ is homeomorphic to βN .

Corollary 1. **There does not exist an extremally disconnected bicom pactum of cardinality c .***

Corollary 2. *The cardinality of an arbitrary infinite closed subspace of an extremally disconnected bicom pactum is not less than 2^c .*

Problem. *Characterize those bicom pacta which are homeomorphic to subspaces of extremally disconnected bicom pacta.*

Let X be a fixed topological space. By a **pair** we shall mean an expression of the form (x, D) , where D is some countable discrete subspace of the space X , and $x \in X$ is a point from the closure of the set D (not belonging to D). The pairs $(x, D), (x', D')$ are called **equivalent**, $(x, D) \approx (x', D')$, if the spaces $\{x\} \cup D$ and $\{x'\} \cup D'$ are homeomorphic (the sets $\{x\} \cup D$ and $\{x'\} \cup D'$ are supposed to be endowed with the topology induced from X). A pair (x^*, D^*) will be called a **subpair** of the pair (x, D) (notation $(x^*, D^*) < (x, D)$), if $x = x^*$ and $D^* \subseteq D$. The pairs (x, D) and (x', D') are called **weakly equivalent** (written as $(x, D) \sim (x', D')$) if they have equivalent subpairs.

Proposition 2.* *Let X be a separable extremally disconnected space and let $x_0 \in X$ be any point. Denote by $\mathfrak{F}$ the family—*

* All spaces considered below are assumed in advance to be regular T_1 -spaces.

** By c throughout this paper is denoted the cardinality of the continuum.

*** Proposition 2 generalizes the corresponding assertion of Z. Frolík on βN , whose proof was kindly communicated by him to the author and formed the basis of the argument given below.

of all pairs of the form (x_0, D) , and let $\mathcal{E}$ be any subfamily of the family $\mathcal{F}$, consisting of pairwise not weakly equivalent pairs. Then the cardinality of the family $\mathcal{E}$ does not exceed c .

Proof. To each pair $(x_0, D) \in \mathcal{E}$ assign, in some way, a countable disjoint cover $\gamma(x_0, D)$ of the set D by open-and-closed subsets of X . Denote by S some countable everywhere dense subset of X . It is clear that the system $\gamma(x_0, D)$ is uniquely recovered from the system $\gamma'(x_0, D)$, formed by the intersections of the elements of the system $\gamma(x_0, D)$ with the set S . The cardinality of all distinct systems $\gamma'(x_0, D)$ does not exceed c , since the cardinality of the set of all possible partitions of a countable set into a countable collection of pairwise disjoint sets is equal to c .

Suppose that the cardinality of the family $\mathcal{E}$ is greater than c . Then there are two pairs $(x_0, D_1), (x_0, D_2)$ in $\mathcal{E}$ for which $\gamma'(x_0, D_1) = \gamma'(x_0, D_2)$. In that case also $\gamma(x_0, D_1) = \gamma(x_0, D_2)$. Since $(x_0, D_1) \approx (x_0, D_2)$, $[D_1 \cap D_2] \neq x_0$. Each element of the system $\gamma(x_0, D_1) = \gamma(x_0, D_2)$ meets the set D_1 in exactly one point and the set D_2 in exactly one point. Those elements of the system γ for which these points do not coincide we enumerate, in some way, in a sequence $\mathcal{U}_1, \mathcal{U}_2, \dots, \mathcal{U}_n, \dots$, and put $x_n = \mathcal{U}_n \cap D_1$, $y_n = \mathcal{U}_n \cap D_2$.

By what has been said, $x_n \neq y_n$, $n = 1, 2, \dots, \dots \infty$. From $[D_1 \cap D_2] \neq x_0$ it follows that

$$x_0 \in \left[\bigcup_{n=1}^{\infty} \{x_n\} \right] \quad \text{and} \quad x_0 \in \left[\bigcup_{n=1}^{\infty} \{y_n\} \right].$$

For each n choose, around the points x_n, y_n , disjoint open neighborhoods Ox_n, Oy_n , contained in $\mathcal{U}_n$.

Then: 1)

$$\left[\bigcup_{n=1}^{\infty} O x_n \right] \ni x_0;$$

2)

$$\left[\bigcup_{n=1}^{\infty} O x_n \right] \cap \left\{ \bigcup_{n=1}^{\infty} O y_n \right\};$$

3)

$$\left[\bigcup_{n=1}^{\infty} O y_n \right] \ni x_0.$$

These three relations are in contradiction with the extremal disconnectedness of the space X . Proposition 2 is proved.

Let now P be an arbitrary bicomact infinite subspace of the extremally disconnected space X , A some countable discrete subspace of the space P (fixed in what follows), and x_0 a point of the closure of the set A , $x_0 \notin A$. Consider the family $\mathcal{F}$ of all (x_0, D) , where $D \subseteq P$, and let $\mathcal{E}$ be such a subfamily of the family $\mathcal{F}$ that: 1) every pair in $\mathcal{F}$ is weakly equivalent to some pair in $\mathcal{E}$; 2) any two distinct pairs in $\mathcal{E}$ are not weakly equivalent. A family $\mathcal{E}$ with the indicated properties exists by virtue of the axiom of choice. From Proposition 2 it follows that the cardinality of the family $\mathcal{E}$ does not exceed c . Consider all subpairs of pairs belonging to $\mathcal{E}$; their totality, whose cardinality is obviously equal to c , we denote by $\mathcal{B}$. Finally, denote by Q the family of those pairs (y, D) such that: 1) $y \in [A] \setminus A$; 2) $D \subseteq A$; 3) the pair (y, D) is equivalent to some pair belonging to the family $\mathcal{B}$.

Proposition 3. The cardinality of the family Q is not greater than c .

Proof. For each pair $(y, D) \in Q$ choose, in some way, a pair $(x_0, D_0) \in \mathcal{B}$ equivalent to it and a homeomorphism $f : \{x_0\} \cup D_0 \rightarrow \{y\} \cup D$. It is clear that $f x = y$, for x and y are the only non-isolated points of the spaces $\{x_0\} \cup D_0$ and $\{y\} \cup D$. Hence f induces a certain one-to-one mapping $\hat{f} : D_0 \rightarrow D$.

The following is obvious.

Lemma 2. If for homeomorphisms $f : \{x_0\} \cup D_0 \rightarrow \{y\} \cup D$ and $f' : \{x_0\} \cup D_0 \rightarrow \{y'\} \cup D$ the induced mappings $\hat{f}$ and $\hat{f}'$ coincide, then $f = f'$ (i.e. $y = y'$).

Consider the families $T = \{(x_0, D) \in \mathcal{B}\}$ and $\Pi = \{D^* : D^* \subseteq A\}$. Lemma 2 means that the cardinality of the set Q is not greater than the cardinality of the set of all one-to-one mappings of the elements of the family T onto elements of the family Q . It is clear that for any fixed $D_0 \in T$ and $D_0^* \in \Pi$ there exist no more than c distinct one-to-one mappings of the set D_0 onto the set D_0^* (for D_0 and D_0^* are countable sets).

But the cardinality of $T \leq c$ and the cardinality of $\Pi \leq c$. Hence also the cardinality of the set of all possible pairs of elements of the families Π and T

does not exceed c . It follows that the cardinality of $Q \leq c$. Proposition 3 is proved.

The set $[A] \setminus A$, as the remainder of the Čech extension of a countable discrete set, has cardinality 2^c . Denote by Q^1 the set of first elements of the pairs of the family Q . By Proposition 3, the cardinality of $Q^1 \leq c$; therefore the set $([A] \setminus A) \setminus Q^1$ is nonempty. We shall show that no point $x^* \in ([A] \setminus A) \setminus Q^1$ can be carried to the point x_0 by a homeomorphism of the space P onto the space P .

Suppose the contrary; let $g : P \rightarrow P$ be a homeomorphism of the space P onto the space P (possibly, onto a subspace P), with $gx^* = x_0$. Since A is a discrete subspace of the space P , the set gA , which we denote by A_0 , is a discrete subspace of the space X . We have $x_0 \in [A_0] \setminus A_0$, since $gx^* = x_0$ and $x^* \in [A] \setminus A$. Thus (x_0, A_0) , (x^*, A) are pairs, and $(x_0, A_0) \in \mathcal{F}$, and $(x_0, A_0) \approx (x^*, A)$. By the definition of the family $\mathcal{E}$, the pair (x_0, A_0) is weakly equivalent to some pair belonging to the family $\mathcal{E}$. This means that some subpair (x_0, D_0) of the pair (x_0, A_0) is equivalent to some pair belonging to the family $\mathcal{B}$. From $(x_0, A_0) \approx (x^*, A)$ it follows, obviously, that the subpair (x_0, D_0) of the pair (x_0, A_0) is equivalent to some subpair (x^*, D) of the pair (x^*, A) . Since the relation $\approx$ is transitive, we can conclude that the pair (x^*, D) is equivalent to some pair belonging to the family $\mathcal{B}$. But then $(x^*, D) \in Q$ —contrary to the fact that $x^* \notin Q^1$.

Definition 1. A subspace Y of a space X is called **homogeneous in X** if for any two points x, y of Y there exists a homeomorphism f from the space Y into X such that $fx = y$.

Above it was proved

Main Proposition 4. An infinite bicomact subspace P of an extremally disconnected separable space X is not homogeneous in X .

Here is an extremely useful obvious assertion, distinguishing the concept of “homogeneity in” from ordinary homogeneity.

Proposition 5. If $X \supset Y \supset Z$ and Y is homogeneous in X , then Z is homogeneous in X .

From Main Proposition 4 and Proposition 5 it follows immediately that

Main Theorem 1. If Y is a subspace homogeneous in X of a separable extremally disconnected space X , then the k -leader of the space Y is discrete (see (2)).

Corollary 1. A separable extremally disconnected homogeneous k -space is discrete and countable.

Since the preimage of a separable k -space under an irreducible perfect mapping is a separable k -space (see (2)), Theorem 1 implies

Corollary 2. The absolute (see (3)) of a nondiscrete separable k -space is nonhomogeneous.

Corollary 3. The absolute of a separable nondiscrete p -space (see (4)) is nonhomogeneous.

Corollary 4. The absolute of a separable nonmetrizable space is nonhomogeneous.

Here by a nonmetrizable space is meant any space not representable as the sum of a countable family of its locally zero-dimensional (in the sense of ind) subspaces. In this case the natural mapping of the absolute onto the space cannot be finite-to-one. Now Corollary 2 from Proposition 1 and Propositions 4 and 5 are applied.

Theorem 2. If Y is a subspace homogeneous in X of an extremally disconnected space X of weight c , then the k -leader (see (2)) of the space Y is discrete.

Lemma 3. Every extremally disconnected space X of weight τ is embedded in the absolute of the space D^τ .

We may assume that X lies in D^τ . Therefore it is enough to prove the following Lemma 4.

Lemma 4. If $X \subset \widetilde{X}$, where X is extremally disconnected, and a perfect mapping $f : Z \rightarrow \widetilde{X}$ is given, then the space X is homeomorphic to some subspace of the space Z .

Indeed, consider $Z_1 = f^{-1}X$ and $f_1 = f|_{Z_1}$, $f_1 : Z_1 \rightarrow X$. The mapping f_1 is perfect; therefore there exists a closed set Z_2 in Z_1 such that $fZ_2 = X$ and the mapping $f_2 = f|_{Z_2}$ is irreducible. Since X is extremally disconnected, Z_2 is homeomorphic to X —Lemma 2 is proved.

We now prove Theorem 2. Applying Lemmas 3 and 4, we conclude that X is homeomorphic to a subspace of the absolute of the space D^c . But D^c , and hence its absolute, is separable. Now Y is a subspace of an extremally disconnected separable space. We can apply Theorem 1. Theorem 2 is proved.

Corollary 1. An extremally disconnected homogeneous k -space of weight c is discrete.

Corollary 2. An infinite extremally disconnected bicomactum of weight c is nonhomogeneous. Any infinite bicomactum containing one is also nonhomogeneous.

Remark. Since βN is the absolute of the simplest infinite compactum, the nonhomogeneity of the remainder $\beta N \setminus N$ is a simple special case of Theorem 1. The first proofs of the nonhomogeneity of $\beta N \setminus N$ were based on the continuum hypothesis (see, in particular, (5)); later Z. Frolík showed how to dispense with the latter.

The answers to the following questions are unknown to the author:

- 1°. Does there exist an infinite homogeneous extremally disconnected bicom-
pactum?
- 2°. Can a nondiscrete extremally disconnected space be a group space?
- 3°. Is it true that the absolute of an arbitrary bicom-compactum of weight c is
homogeneous?
- 4°. For which spaces X is the supremum of the cardinalities of the closures of
discrete subspaces of the space X equal to the cardinality of X ?

Moscow State University
named after M. V. Lomonosov

Received
5 X 1966

REFERENCES

1. M. H. Stone, Trans. Am. Math. Soc., **41**, 375 (1937).
2. A. V. Arhangel'skii, Tr. Mosk. matem. obshch., **13**, 3 (1965).
3. V. I. Ponomarev, DAN, **149**, 26 (1963).
4. A. V. Arhangel'skii, Matem. sborn., **67** (109), 1, 55 (1965).
5. L. Gillman, M. Jerison, *Rings of Continuous Functions*, Princeton, 1960.

* This fact was discovered by V. I. Ponomarev and me in a friendly conversation.

Note: Figure translations are in progress. See original paper for figures.

*Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the
original.*