

ON ARITHMETIC SUMS OF RECURSIVELY PROJECTIVE SETS

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Abstract

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MATHEMATICS

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ON ARITHMETIC SUMS OF RECURSIVELY PROJECTIVE SETS

(Presented by Academician P. S. Novikov on 14 IV 1966)

§ 1. Introduction

In set theory, an important role is played by the study of operations by means of which sets of various classes are obtained. The question of the role of the operation of arithmetic summation in set theory was posed by P. S. Novikov.

The **arithmetic sum** of two sets E_1 and E_2 is the set of all numbers of the form $x_1 + x_2$, where $x_1 \in E_1$, $x_2 \in E_2$. We shall denote it by $E_1 \oplus E_2$. From the geometric point of view, $E_1 \oplus E_2$ is the projection onto the axis Ox , in the direction of the lines $x + y = \text{const}$, of the Cartesian product $E_1 \times E_2$.

For sets of Euclidean space, B. S. Sodyomov obtained the following results: in the class of sets T , containing all intervals and closed under countable summation, taking complements, and arithmetic summation, one can find representatives of arbitrarily high classes of projective sets of N. N. Luzin. The question remained open whether the class T coincides with the class of all projective sets or not.

As is known, there are a number of analogies between the projective sets of N. N. Luzin and sets of natural numbers of the recursively projective classes of Kleene–Mostowski (arithmetical sets in the sense of Gödel). After Sodyomov's works, there naturally arose the question of arithmetic summation of recursively projective sets.

In connection with arithmetic sums of recursively projective sets, it is natural to consider sets of integers, since the arithmetic sum of recursive sets of natural numbers does not lead outside the class of recursive sets. The recursive theory of integers is constructed in the same way as the recursive theory of natural numbers. In all definitions one need only bear in mind that the elements of all sets, tuples, etc., under consideration may be integers, both positive and negative.

Using special methods of the theory of recursively projective sets, it has been possible to obtain not only results analogous to Sodyomov's results in the theory

of projective sets of N. N. Luzin, but also stronger ones.

First, in the class of sets containing the recursive sets and closed under the operations of arithmetic summation and taking complements, there are representatives of arbitrarily high classes of recursively enumerable sets.

Second, the class of sets containing all recursive sets and closed under the operations of finite intersection, arithmetic summation, and taking complements coincides with the class of recursively projective sets.

§ 2. The class of symmetrically dense enumerable sets

As the most interesting case we shall consider one-to-one arithmetic sums, i.e., such sums $E_1 \oplus E_2 = E$ that if $x \in E$, then x can be represented uniquely as $a+b$, where $a \in E_1$, $b \in E_2$. It is easy to show that in this case every enumerable set decomposable into a nonrepeating arithmetic sum of two recursive sets is represented as a union of disjoint congruent recursive sets forming a recursive sequence.

In the subsequent theorems of this section we shall consider only nonrepeating arithmetic sums. We shall denote enumerable sets by P -s., and recursive sets by R -s.

Let us introduce several notions. A **tuple** will mean a finite ordered sequence of integers; a **segment** is a tuple composed of numbers following one immediately after another in the sequence of integers; the **length of a segment** is the number of elements of the set constituting the segment. A segment is **contained in a set** if every element of the segment belongs to that set. Two segments will be called **symmetric segments** if each of them is composed of the numbers of the other, taken with opposite signs. A set will be called **symmetrically dense** (s.d.) if it contains symmetric segments of arbitrarily large length.

Theorem 1. *Every s.d. P -s. is decomposable into a nonrepeating arithmetic sum of two R -s.*

The question arises of the existence of s.d. P -s. It turns out that the class of s.d. P -s. contains representatives of all known classes of P -s.

Theorem 2. *There exist s.d. P -s.*

Theorem 3. *There exist s.d. creative sets.*

It is easy to show that every hyperimmune set is s.d.

Theorem 4. *Every simple set is s.d.*

Theorem 5. *There exist s.d. P -s. which are neither recursive, nor creative, nor simple, nor hyperimmune.*

From Theorems 1-5 there follow the following corollaries:

Corollary 1. *There exist R -s. decomposable into the arithmetic sum of two infinite R -s.*

In particular, the set of integers is decomposable into the arithmetic sum of two infinite R -s.

Corollary 2. *There exist creative sets decomposable into a nonrepeating arithmetic sum of two infinite P -s.*

Corollary 3. *Every simple and every hyperimmune set is decomposable into a nonrepeating arithmetic sum of two infinite R -s.*

Corollary 4. *There exists a P -s. which is neither recursive, nor creative, nor simple, nor hyperimmune, decomposable into the arithmetic sum of two infinite R -s.*

The property of symmetric density is a sufficient condition for decomposability of a P -s. into the arithmetic sum of two R -s. However, it is not necessary.

Theorem 6. *There exist P -s. which are not symmetrically dense, decomposable into a nonrepeating arithmetic sum of R -s.*

An example may be a P -s. that is the intersection of an s.d. P -s. and the set of even integers.

Theorem 7. *There exists a P -s. not decomposable into the arithmetic sum of two infinite R -s.*

An example may be any P -s. consisting only of positive numbers.

We shall call a set **absolutely indecomposable** if it is not decomposable into the arithmetic sum of two sets, each of which contains more than one element.

Theorem 8. *There exist absolutely indecomposable sets.*

An example may be any set whose elements are integers of the same parity, together with one element of the other parity.

Theorem 9. *Every P -s. can be represented as the intersection of two s.d. P -s.*

It follows from this that the class of P -sets can be obtained from the class of R -sets by means of finite intersection and arithmetic summation.

§ 3. Separation theorems in the class of c.d. P -sets. In the class of c.d. P -sets there are separation theorems analogous to the separation theorems for all enumerable sets and their complements*. The basis of this class will be the symmetrically dense recursive sets (c.d. R -sets):

Theorem 10. *Any two disjoint complements of e.s. R -sets are separable by sets of the basis.*

Theorem 11. *There exist two disjoint e.s. P -sets not separable by sets of the basis.*

§ 4. On arithmetic sums of recursively projective sets of higher classes.

By definition, sets of the second projective class P_2 -sets are obtained as projections, onto the X -axis, of sets complementary to P -sets (cP -sets). Sets of the n -th projective class (P_n -sets) are obtained as projections, onto the X -axis, of sets complementary to P_{n-1} -sets (cP_{n-1} -sets).

If one starts from the sets of the basis, then, by means of the operations of arithmetic summation and taking complements, one can obtain recursively projective sets of any class.

Theorem 12. *Every e.s. P_2 -set containing an e.s. P -set is decomposable into the arithmetic sum of a P -set and a cP -set.*

Theorem 13. *Every P_n -set containing an e.s. P -set is decomposable into the arithmetic sum of an R -set and a cP_{n-1} -set.*

The class of sets containing all R -sets and closed under the operations of finite intersection, arithmetic summation, and taking complements coincides with the class of all recursively projective sets.

This fact rests on the following theorem:

Theorem 14. *Every P_n -set can be represented as the intersection of two sets of the same class, each of which contains an e.s. P -set.*

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* Separation and inseparability theorems for enumerable sets are presented, for example, in (3).

Note: Figure translations are in progress. See original paper for figures.

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