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Abstract

Full Text

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MATHEMATICS

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INTERPOLATION FAMILIES OF FUNCTIONS AND EMBEDDINGS OF SETS IN EUCLIDEAN AND PROJECTIVE SPACES

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1. Let natural numbers m and $k \leq m$ be given. An **interpolation family of functions of order m and degree k** , or an **(m, k) -system**, will mean a collection of real functions

$$f_0(t), f_1(t), \dots, f_m(t), \quad (1)$$

continuous on some topological space T and satisfying the following condition.

(A). For any distinct points $t_0, t_1, \dots, t_k$ of the space and any real numbers $a_0, a_1, \dots, a_k$, there exists a polynomial in the system (1)

$$p(t) = \sum_{i=0}^m c_i f_i(t), \quad (2)$$

such that $p(t_j) = a_j$ ($j = 0, 1, \dots, k$).

It is easy to see that condition (A) in this definition may be replaced by either of the following:

(B). For any distinct points $t_0, t_1, \dots, t_k$ of the space T , the rank of the matrix $\|f_i(t_j)\|_{i=0}^{j=k}$ is equal to $k + 1$.

(C). Any $m - k + 1$ linearly independent polynomials of the form (2) have on T no more than k common zeros.

As a rule, we shall assume the space T to be a bicomact Hausdorff space (a bicomactum).

Interpolation families of functions, besides being of independent interest, also arise in the problem of best approximation of continuous functions by generalized polynomials ^(1,2), by generalized rational functions ⁽³⁾, in the finite moment

problem ⁽⁴⁾, and in the problem of finite-dimensional linear positive operators ⁽⁵⁾.

2. A subset X of the projective space P^m is called k -regular if every $(k-1)$ -dimensional (projective) plane in P^m meets this set in at most k points. We shall call any homeomorphic mapping of a topological space T onto a k -regular subset X of P^m a k -regular embedding. A k -regular subset of the Euclidean space E^m and a k -regular embedding into this space are defined analogously.

If an (m, k) -system of functions (1) is given on a bicompactum T , then, by assigning to each point $t \in T$ the point of projective space P^m with homogeneous coordinates $f_0(t), f_1(t), \dots, f_m(t)$, we obtain a k -regular embedding $F: T \rightarrow P^m$. Conversely, if there is some k -regular embedding of the space T in P^m , then the homogeneous coordinates of a variable point of the image give an (m, k) -system of functions defined on T .

Sometimes one has to consider interpolation systems (1) for which $f_0(t) \equiv 1$. Such systems define a k -regular embedding of the space T in the Euclidean space E^m .

It follows from what has been said, in particular, that a bicompactum on which an (m, k) -system of functions exists is metrizable (i.e. is a compactum) and has finite dimension, which, as can be shown, does not exceed $m - k + 1$.

3. Denote by $D_k(T)$ (respectively, by $d_k(T)$) the least dimension of a projective (respectively, Euclidean) space into which the compactum T is k -regularly embeddable. Clearly, $D_k(T) \leq d_k(T)$. Apparently, the question of computing $D_k(T)$ for $k > 1$ has not previously been considered. As for the number $d_k(T)$, its exact value is known only in a few cases. In papers ⁽⁶⁻¹⁰⁾, upper and lower estimates for it were obtained for various classes of compacta. Here we indicate new estimates of the numbers $D_k(T)$ and $d_k(T)$ for certain spaces T . As consequences, some relations are obtained between the orders and degrees of interpolation systems of functions on these spaces.

The lower estimates for the number $D_k(T)$ are based on the following theorem.

Theorem 1. *Let a compactum T be k -regularly embedded in the projective space P^m , and let $t_1, t_2, \dots, t_i$ be distinct points of this compactum, their number being $i = [(k+1)/2]$. Then there exist closed neighborhoods $M_1, M_2, \dots, M_i$ of these points relative to T , mutually disjoint and such that the direct product*

$$M_1 \times M_2 \times \dots \times M_i \times S,$$

where S is a simplex of dimension $[k/2]$, is homeomorphically embeddable in the Euclidean space E^m .

Theorem 2. *For the n -dimensional cube I^n the inequality*

$$D_k(I^n) \geq [(k+1)/2](n-1) + k$$

holds.

Theorem 3. *If a topological space contains an n -dimensional cube, then on it there cannot exist an (m, k) -system of even degree k satisfying the inequality*

$$k > 2m/(n+1),$$

or of odd degree k , satisfying the inequality

$$k > 2m/(n+1) - (n-1)/(n+1).$$

We shall call an n -dimensional umbrella a set T_n lying in the space E^{n+1} with coordinates $(t_1, t_2, \dots, t_{n+1})$ and consisting of the n -dimensional ball

$$t_1^2 + t_2^2 + \dots + t_n^2 \leq 1, \quad t_{n+1} = 0$$

and the segment

$$t_1 = t_2 = \dots = t_n = 0, \quad 0 \leq t_{n+1} \leq 1,$$

as well as any set homeomorphic to T_n .

Theorem 4. *The direct product of p umbrellas of dimension n is not topologically embeddable in the Euclidean space E^{np+p-1} .*

For $n = 1$ this assertion was proved in ⁽⁹⁾.

Theorem 5. *If a finite-dimensional compactum T contains p umbrellas of dimension n and if $p \leq [(k+1)/2]$, then the inequality*

$$D_k(T) \geq [(k+1)/2](n-1) + k + p.$$

holds.

Theorem 6. *Let a topological space contain p umbrellas of dimension n . Then on it there cannot exist (m, k) -systems of even degree k , satisfying the inequalities*

$$k \geq 2p, \quad k > 2(m-p)/(n+1),$$

or of odd degree k , satisfying the inequalities

$$k \geq 2p-1, \quad k > 2(m-p)/(n+1) - (n-1)/(n+1).$$

4. We indicate an upper estimate for the number $d_{2k}(E^n)$. The derivation of this estimate is based on the following geometric theorem of Radon ⁽¹¹⁾:
every set of

$m + 2$ points lying in an m -dimensional linear space can be divided into two nonempty nonintersecting subsets whose convex hulls have a common point. It follows from Radon's theorem that the system of continuous functions

$$f_0(t) = 1, \quad f_1(t), \dots, f_m(t), \quad (3)$$

defined on the space T , will be an interpolation system of degree $2k$ in the case when it has the following property:

(Γ_k) . For any s ($s \leq k$) points of the space T there exists a polynomial in the system (3) that is equal to zero at these points and positive on their complement.

Indeed, in the contrary case, in the space E^m there is a plane E^{2k-1} containing distinct points

$$x_1, x_2, \dots, x_{2k+1} \quad (4)$$

of the image $F(T)$ of the set T under the mapping determined by the system (3). By Radon's theorem the set of points (4) can be represented (with a suitable choice of the numbering of these points) as the union of two nonintersecting subsets $\{x_1, \dots, x_s\} \cup \{x_{s+1}, \dots, x_{2k+1}\}$, $s \leq k$, whose convex hulls intersect. On the other hand, by virtue of the property (Γ_k) , in the space E^m there exists a hyperplane E^{m-1} supporting the set $F(T)$ and such that $E^{m-1} \cap F(T) = \{x_1, \dots, x_s\}$.

Returning to the convex hulls of the sets $\{x_1, \dots, x_s\}$ and $\{x_{s+1}, \dots, x_{2k+1}\}$, we see that the first of them lies in the hyperplane E^{m-1} , while the second lies in one of the open half-spaces determined by this hyperplane. Consequently, these convex hulls cannot intersect.

Consider on the space E^n the system of functions

$$\begin{aligned} &1, t_i, t_{i_1} t_{i_2}, t_{i_1} t_{i_2} t_{i_3}, \dots, t_{i_1} t_{i_2} \dots t_{i_k}, \\ &t_{i_1} t_{i_2} \dots t_{i_{k-1}} (t_1^2 + t_2^2 + \dots + t_n^2), t_{i_1} t_{i_2} \dots t_{i_{k-2}} (t_1^2 + t_2^2 + \dots + t_n^2)^2, \dots, \\ &\dots, t_{i_1} t_{i_2} (t_1^2 + t_2^2 + \dots + t_n^2)^{k-2}, t_i (t_1^2 + t_2^2 + \dots + t_n^2)^{k-1}, \\ &(t_1^2 + t_2^2 + \dots + t_n^2)^k, \end{aligned} \quad (5)$$

where $(t_1, t_2, \dots, t_n)$ are the coordinates of the variable point of this space and each index $i, i_1, i_2, \dots, i_k$, independently of the others, takes the values $1, 2, \dots, n$. This system of functions has the property (Γ_k) . Indeed, if $\rho(x, y)$ is the metric in E^n and $x_1, x_2, \dots, x_k$ are arbitrary points of this space, then the product

$$\rho^2(x, x_1)\rho^2(x, x_2) \cdots \rho^2(x, x_k)$$

is precisely the polynomial in the system (5) whose existence is required by the property (Γ_k) . Thus the constructed system of functions is interpolation and has degree $2k$. The order of this system, as is easy to see, is equal to the number $C_{n+k}^k + C_{n+k-1}^{k-1} - 1$. Hence,

$$d_{2k}(T) \leq C_{n+k}^k + C_{n+k-1}^{k-1} - 1,$$

where T is an n -dimensional Euclidean space or any of its subsets.

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