

ON THE GLOBAL STRUCTURE OF ANALYTIC SEMI-REDUCIBLE RIEMANNIAN SPACES

A. S. Solodovnikov

1967

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196701.19606>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

UDC 513.838

MATHEMATICS

A. S. SOLODOVNIKOV

ON THE GLOBAL STRUCTURE OF ANALYTIC SEMI-REDUCIBLE RIEMANNIAN SPACES

(Presented by Academician I. G. Petrovskii, February 6, 1967)

1°. Various questions, relating mainly to mappings of Riemannian spaces, lead to the so-called **semi-reducible spaces**, i.e., Riemannian spaces with a metric of the form

$$ds^2 = d\rho^2 + f d\theta^2. \quad (1)$$

In this notation it is assumed that the coordinates $x_1, \dots, x_n$ are divided into two groups, say $x_1, \dots, x_k$ and $x_{k+1}, \dots, x_n$, with the metric $d\rho^2$ and the function f depending on the coordinates of the first group, while $d\theta^2$ depends on the coordinates of the second.

In the works known to us, semi-reducible spaces are studied mainly locally. In the present article the spaces are considered globally, and the metric is assumed to be analytic. More precisely, we consider a space V_n satisfying the following condition:

Condition A. V_n is a connected complete analytic Riemannian space whose metric, in some coordinate domain U ,

$$|x_i - x_i^0| < \varepsilon \quad (i = 1, \dots, n)$$

is represented in the form (1), where

$$d\rho^2 = \sum_{\alpha, \beta=1}^k a_{\alpha\beta}(x_1, \dots, x_k) dx_\alpha dx_\beta, \quad d\theta^2 = \sum_{\gamma, \delta=k+1}^n b_{\gamma\delta}(x_{k+1}, \dots, x_n) dx_\gamma dx_\delta$$

are positive definite quadratic forms with analytic coefficients, and $f(x_1, \dots, x_k)$ is a positive analytic function.

Fig. 1.

Figure 1: Fig. 1.

The question that will interest us in this article is to what extent the decomposition (1) extends to the whole space V_n , and how V_n is structured globally.

2°. We indicate two types of examples of spaces V_n satisfying condition A.

- 1) Let V_k and V_{n-k} be two connected complete analytic Riemannian spaces, and let F be a strictly positive analytic function on V_k . Then one may construct a new space of dimension n by defining it as the Cartesian product of V_k and V_{n-k} , endowed with the metric

$$d\rho^2 + Fd\theta^2, \quad (2)$$

where $d\rho^2$ is the metric in V_k , and $d\theta^2$ is the metric in V_{n-k} . We shall denote this space in what follows by $V_k \times FV_{n-k}$. Obviously, it satisfies condition A.

- 2) In the $(n+1)$ -dimensional Euclidean space E_{n+1} , fix two subspaces, E_k and E_{k+1} , with $E_k \subset E_{k+1}$. Let σ be a k -dimensional surface in E_{k+1} , analytic, regular, connected, and complete (with respect to the induced metric); assume also that this surface is symmetric with respect to E_k . Rotating σ in E_{n+1} about the subspace E_k , we obtain some n -dimensional surface S . From obvious

from geometric considerations it follows that the metric on S has the form (2), where $d\rho^2$ is the metric in σ , $d\theta^2$ is the metric of the sphere S_{n-k} of unit radius, and $F = \varphi^2$, where φ is the distance from an arbitrary point $p \in S$ to E_k . The space S satisfies condition A. We also note the inequality $d\rho \geq |d\varphi|$, which follows from considering the surface σ .

Fig. 1.

The construction of the surface S admits the following generalization.

Let V_k be a connected complete analytic Riemannian space and let F be an analytic function on it having the following properties:

- a) $F \geq 0$, and the set $\{x \in V_k \mid F(x) = 0\}$ is a complete geodesic surface* V_{k-1} (generally speaking, not connected), dividing the space into two half-spaces V' and V'' .
- b) There exists an isometry τ of the space V_k , the set of fixed points of which is V_{k-1} ; hence it follows that the points p and $\tau(p)$ ($p \notin V_{k-1}$) belong to different half-spaces.
- c) $F(p) = F(\tau(p))$ for all $p \in V_k$.

d)

$$\lim_{p \rightarrow p^*} \frac{F(p)}{r^2(p)} = 1,$$

where p^* is any point of V_{k-1} , and $r(p)$ is the distance from p to V_{k-1} .

In the construction of the surface S , the role of V_k was played by the surface σ , and the role of V_{k-1} by the intersection of σ with E_k .

Any pair V_k, F with the properties listed allows one to define a new space V_n , where n is any integer greater than k . The space V_n is defined by the following construction.

Onto each pair of points $p, \tau(p)$, where $p \in V'$, a sphere of radius $F(p)$ and of dimension $n - k$ is "strung," in such a way that the points p and $\tau(p)$ are its diametrically opposite points. In the resulting set V_n a metric (2) is introduced, where $d\rho^2$ is the metric in V_k , and $d\theta^2$ is the metric on the sphere S_{n-k} of unit radius. As a result V_n becomes a connected complete analytic Riemannian space. This space satisfies condition A. Figure 1 depicts the case $n - k = 1$.

The space V_n defined in the above manner will be called a **space of rotation** and denoted $V_k \circ FS_{n-k}$.

3°. We present some facts and definitions. In each of them the space is assumed to be analytic.

Two spaces V_n and V'_n are called **locally isometric** if some domain in V_n is isometric to some domain in V'_n .

Let K be the class of all connected complete spaces locally isometric to a given space V_n . In the class K there always exists a unique, up to isometry, simply connected space $\widetilde{V}_n$. Any other space of this class is isometric to the quotient space $\widetilde{V}_n/\Gamma$, where Γ is a discrete group of motions of $\widetilde{V}_n$ without fixed points.

A **k -dimensional analytic sheet** in V_n is a pair (E_k, ψ) , where E_k is a k -dimensional Euclidean space, and ψ is its analytic regular mapping into V_n ; it is also required that $\psi(x) \neq \psi(y)$ for $x \neq y$ ($x, y \in E_k$). In what follows we identify the sheet (E_k, ψ) with the set $T = \psi(E_k) \subset V_n$.

The sheet T is called **complete geodesic** if every geodesic in T is also a geodesic in V_n .

A **k -dimensional analytic surface** in V_n is a pair (M, Ψ) , where M is a k -dimensional analytic manifold, and Ψ is its regular analytic mapping into V_n .

* The necessary definitions are given below in Section 3°.

For each analytic sheet T in V_n there exists a uniquely determined analytic surface $\widetilde{T}$, which it is natural to regard as the **maximal analytic continuation of the sheet T** . We shall indicate how this surface is defined.

Two k -dimensional analytic sheets will be considered as **belonging to the same k -dimensional germ at a point p** , if both sheets pass through p and some of their subsheets*, containing p , coincide (as subsets of V_n). The germ to which the sheet T belongs at the point $p \in T$ will be denoted by $\tau_p(T)$; the set $\{\tau_p(T) \mid p \in T\}$ will be denoted by $g(T)$.

Let G be the set of all k -dimensional germs in V_n . Introduce in G the topology in which the subsets $g(T)$ (where T is an arbitrary k -dimensional sheet) form, by definition, a basis of open sets. This topology is Hausdorff.

Associating with each k -dimensional germ at a point p the point p itself, we obtain a mapping $\pi : G \rightarrow V_n$. Since π maps each $g(T)$ homeomorphically onto T , an analytic structure arises on the space G , as well as an analytic metric borrowed from V_n . Thus G is an analytic Riemannian space.

For each sheet T we now define $\tilde{T}$ as the connected component of the space G containing $g(T)$. It is easy to see that $(\tilde{T}, \pi)$ is an analytic surface containing, in the natural sense, the sheet T .

The question of the completeness of the surface $\tilde{T}$ (from the point of view of the induced metric) is very important. In this connection we shall confine ourselves only to the following propositions.

If V_n is a complete space and T is a completely geodesic sheet in it, then the surface $\tilde{T}$ is complete.

4°. Let V_n satisfy condition A. Through a point $p^0(x_1^0, \dots, x_n^0) \in U$ pass two mutually orthogonal coordinate sheets: the k -dimensional sheet $T^{(\rho)} : x_{k+1} = x_{k+1}^0, \dots, x_n = x_n^0$, carrying the metric $d\rho^2$, and the $(n-k)$ -dimensional sheet $T^{(\theta)} : x_1 = x_1^0, \dots, x_k = x_k^0$, whose metric is $f(x_1^0, \dots, x_k^0) d\theta^2$. In view of the obvious inequality $ds^2 \geq d\rho^2$, valid throughout the domain U , the sheet $T^{(\rho)}$ is completely geodesic.

Lemma 1 (fundamental). The surfaces $\tilde{T}^{(\rho)}$ and $\tilde{T}^{(\theta)}$ (the maximal analytic continuations of the sheets $T^{(\rho)}$ and $T^{(\theta)}$) are complete.

Let us note that the completeness of $\tilde{T}^{(\rho)}$ follows directly from the last proposition of item 3°. For the surface $\tilde{T}^{(\theta)}$ the reasoning is more delicate.

Lemma 2. If the space V_n is simply connected, then the function f admits a single-valued analytic continuation F to the whole space $\tilde{T}^{(\rho)}$; moreover $F \geq 0$.

Theorem 1. If V_n is simply connected and $F \neq 0$ everywhere in $\tilde{T}^{(\rho)}$, then V_n is isometric to

$$\tilde{T}^{(\rho)} \times \frac{F}{c} \tilde{T}^{(\theta)},$$

where $c = f(x_1^0, \dots, x_k^0)$.

Theorem 2. If V_n is simply connected and the set of zeros of the function F is nonempty, then $\tilde{T}^{(\theta)}$ is isometric to a sphere of some radius c (of dimension

$n - k$), and V_n is isometric to the space of rotation

$$\tilde{T}^{(\rho)} \circ c^2 F S_{n-k}.$$

From Theorems 1 and 2 it follows, in particular, that if $k = 1$, then V_n is homeomorphic (in the simply connected case) to one of the spaces

$$E_1 \times \tilde{T}^{(\theta)}, \quad E_n, \quad S_n.$$

In conclusion, the author expresses profound gratitude to P. K. Rashevskii, in whose seminar this work arose.

Moscow State Correspondence
Pedagogical Institute

Received
1 XII 1966

* If a sheet is defined by a pair (E_k, Ψ) , then a subsheet is a pair (D, Ψ') , where D is a domain in E_k , and ψ' is the restriction of ψ to D .

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.