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A. I. Guseinov, M. A. Abduragimov

1967

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Abstract

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UDC 517.948

MATHEMATICS

A. I. GUSEINOV, M. A. ABDURAGIMOV

ON A NONLINEAR BOUNDARY-VALUE PROBLEM

(Presented by Academician I. N. Vekua, 27 IV 1966)

Let L be a simple smooth closed contour dividing the complex plane into the interior domain D^+ and the exterior domain D^- ; suppose that the origin is enclosed by the contour L .

Denote by E^+ (E^-) the space of functions analytic in D^+ (D^-) and continuous in $\overline{D}^+$ ($\overline{D}^-$), with norm

$$\|\Phi^+(z)\| = \max_{z \in \overline{D}^+} |\Phi^+(z)|;$$

respectively

$$\|\Phi^-(z)\| = \max_{z \in \overline{D}^-} |\Phi^-(z)|.$$

Definition 1. We shall say ⁽³⁾ that a function $\omega(s)$ ($0 < s \leq l$) belongs to the class Φ^* if it satisfies the conditions:

- 1) $\omega(s)$ increases monotonically and is finite everywhere;
- 2) $\omega(s) \neq 0$, $\lim_{s \rightarrow 0} \omega(s) = 0$;
- 3) there exists a constant $\tilde{c} > 1$ such that

$$1 < \lim_{s \rightarrow 0} \frac{\omega(cs)}{\omega(s)} \leq \overline{\lim}_{s \rightarrow 0} \frac{\omega(cs)}{\omega(s)} < \tilde{c}.$$

Definition 2. $u(t) \in H_k(\omega)$, $t \in L$, if $|u(t)| \leq k$, $|u(t_1) - u(t_2)| \leq k\omega(|t_1 - t_2|)$, $t_1, t_2 \in L$, where $k = \text{const}$, $\omega(s) \in \Phi^*$.

Statement of the problem. It is required to find a function $\Phi^+(z) \in E^+$ and $\Phi^-(z) \in E^-$ satisfying the boundary condition

$$[\Phi^+(t)]^n + F\left(t, \int_L \frac{f[\tau, \Phi^+(\tau)]}{\tau - t} d\tau\right) = G(t)\Phi^-(t), \quad (*)$$

where $n \geq 2$ is an integer, $G(t)$ is a given function of the class $H_k(\omega)$ on L and is nowhere zero on L ; $f(t, u)$ is a function defined for $t \in L$ and $u = \Phi^+(z) \in E^+$, satisfying the condition:

$$|f(t_1, u_1) - f(t_2, u_2)| \leq A[\omega(|t_1 - t_2|) + (u_1 - u_2)],$$

$$\omega(s) \in \Phi^*, \quad 0 < s \leq l, \quad A = \text{const} > 0, \quad (1)$$

$F(t, v)$ is a function defined for $t \in L$ and $v = \int_L \frac{f(\tau, u)}{\tau - t} d\tau$, $u = \Phi^+(z) \in E^+$, satisfying the conditions:

$$|F(t, v)| \leq B_0(1 + |v|^{n-\varepsilon}), \quad 0 < \varepsilon < n; \quad (2)$$

$$|F(t_1, v_1) - F(t_2, v_2)| \leq B_0[(1 + \tilde{v}^{n-\varepsilon})\omega(|t_1 - t_2|) + (1 + \tilde{v}^{n-1-\varepsilon})|v_1 - v_2|], \quad \tilde{v} = \max(|v_1|, |v_2|). \quad (3)$$

If one takes (1) into account, then it can be shown that

$$|F(t, v)| \leq B(1 + |u|^{n-\varepsilon}), \quad (4)$$

$$|F(t_1, v_1) - F(t_2, v_2)| \leq B[(1 + \tilde{u}^{n-\varepsilon})\omega(|t_1 - t_2|) + (1 + \tilde{u}^{n-1-\varepsilon})|su_1 - su_2|], \quad (5)$$

where $B = \text{const} > 0$, $\tilde{u} = \max(|u_1|, |u_2|)$,

$$Su = \int_L \frac{f(\tau, u)}{\tau - t} d\tau.$$

The operator defined in E^+ by the function

$$F\left(t, \int_L \frac{f(\tau, \Phi^+)}{\tau - t} dt\right)$$

will be denoted simply by $FS\Phi^+$. If we make the substitution

$$\Phi_1^+(z) = [\Phi^+(z)]^n, \quad \Phi_1^-(z) = \Phi^-(z)$$

and use the theory of the linear Riemann boundary-value problem (1, 2, 4), then we shall have

$$\Phi^+(z) = e^{\Gamma^+(z)/n} [P_\chi(z) - H^+FS\Phi^+]^{1/n},$$

$$\Phi^-(z) = e^{\Gamma^-(z)} [P_\chi(z) - H^-FS\Phi^+], \quad (6)$$

where

$$\Gamma(z) = \frac{1}{2\pi i} \int_L \frac{\ln[t^{-\chi}G(t)]}{t-z} dt, \quad H\varphi = \frac{1}{2\pi i} \int_L \frac{\varphi(t)}{t-z} e^{-\Gamma^+(t)} dt, \quad z \in D^+ \cup D^-.$$

Suppose that $\Phi^+(z)$ has no branch points in D^+ and that $\chi = \text{ind } G(t) \geq 0$. Then problem (6) actually splits into n problems corresponding to the different branches of the radical, and the equivalence of problems (*) and (6) is obvious; therefore it is enough to prove the solvability of problem (6).

Put $P_\chi(z) \equiv c$ and consider in the space E^+ the operator

$$W\Phi^+ = e^{\Gamma^+(z)/n} [c - H^+FS\Phi^+]^{1/n},$$

where a certain branch is fixed.

Denote by S_R the closed sphere in E^+ of radius R : $\|\Phi^+(z)\| \leq R$; $S_R(N)$ is the set of functions from S_R whose boundary values belong to the class $H_N(\omega)$.

Lemma 1. If the function $f(t, u)$ satisfies condition (1), then the function

$$v(t) = \int_L \frac{f[\tau, u(\tau)]}{\tau - t} d\tau$$

for $u(t) \in H_N(\omega)$ satisfies the condition

$$|v(t_1) - v(t_2)| \leq DA(1 + N)\omega(|t_1 - t_2|),$$

where $D = \text{const}$ does not depend on A and N .

Lemma 2. If the function $f(t, u)$ satisfies condition (1), and the function $F(t, v)$ satisfies conditions (2), (3), or, equivalently, conditions (4), (5), then the operator

$$H^*u = \frac{1}{2\pi i} \int_\Gamma \frac{F \left[\tau, \int_\Gamma \frac{f(\tau_1, u)}{\tau_1 - \tau} d\tau_1 \right]}{\tau - t} e^{-\Gamma^+(\tau)} d\tau$$

is continuous on the set $S_R(N)$.

Proof. Consider the operators

$$Su = \int_L \frac{f[\tau, u(\tau)]}{\tau - t} d\tau, \quad (7)$$

$$S^*v = \int_L \frac{F[\tau, v(\tau)]}{\tau - t} e^{-\Gamma^+(\tau)} d\tau. \quad (8)$$

To prove the validity of the lemma, it is evidently sufficient to prove the continuity of the operators (7) and (8). Let us prove the continuity of the operator Su .

If (1) is taken into account and $u(t) \in H_N(\omega)$, then it is easy to see that there exists a constant $M^* > 0$ such that $|f[\tau, u(\tau)]| \leq M^*$, $\tau \in L$; $|f[\tau_1, u(\tau_1)] - f[\tau_2, u(\tau_2)]| \leq M^*\omega(|\tau_2 - \tau_1|)$, $\tau_2, \tau_1 \in L$. Then there exists a constant $A^* > 0$, independent of M^* , such that

$$Su = \left| \int_L \frac{f[\tau, u(\tau)]}{\tau - t} d\tau \right| \leq M^* A^*. \quad (9)$$

Let

$$\max_{t \in L} |u_n(t) - u(t)| = \varepsilon.$$

We have

$$\begin{aligned} Su_n - Su &= \int_L \frac{f[\tau, u_n(\tau)] - f[\tau, u(\tau)]}{\tau - t} d\tau, \\ |f[\tau, u_n(\tau)] - f[\tau, u(\tau)]| &= \\ &= |f[\tau, u_n(\tau)] - f[\tau, u(\tau)]|^\mu |f[\tau, u_n(\tau)] - f[\tau, u(\tau)]|^{1-\mu} \leq \\ &\leq A_1^\mu \varepsilon^\mu (2M^*)^{1-\mu} \leq \widetilde{M} \varepsilon^\mu \quad (0 < \mu < 1); \\ |f[\tau_1, u_n(\tau_1)] - f[\tau_1, u(\tau_1)] - f[\tau_2, u_n(\tau_2)] + f[\tau_2, u(\tau_2)]| &\leq \\ &\leq 2\mu A_1^\mu \varepsilon^\mu [2M^*\omega(|\tau_2 - \tau_1|)]^{1-\mu} \leq \widetilde{M} \varepsilon^\mu [\omega(|\tau_2 - \tau_1|)]^{1-\mu}, \end{aligned}$$

where $\widetilde{M} = 2A_1^\mu (M^*)^{1-\mu}$, $\tau_1, \tau_2 \in L$.

Then, according to (9),

$$|Su_n - Su| \leq \varepsilon^\mu \widetilde{M} \widetilde{A},$$

where $\widetilde{A} = \text{const}$ does not depend on $\widetilde{M} \varepsilon^\mu$.

The continuity of the operator S^*v is proved analogously.

Lemma 3. Let the function $f(t, u)$ satisfy condition (1), $f(t, v)$ conditions (4), (5), and let $u = \Phi^+(z) \in S_R(N)$. Then the function

$$v(t) = H^+ F S u \equiv \frac{1}{2\pi i} \int_L \frac{F \left\{ \tau, \int_L \frac{f[\tau_1, u(\tau_1)]}{\tau_1 - \tau} d\tau_1 \right\}}{\tau - t} e^{-\Gamma^+(\tau)} d\tau$$

belongs to the class $H_{\tilde{K}}(\omega)$, where $\tilde{K} = 2\tilde{B}BR^{n-1-\varepsilon}\{[R+DA(1+N)]M+rkR\}$, $M = \max(\|e^{\Gamma^+(z)}\|, e^{-\Gamma^-(z)}, 1)$, and $\tilde{B}, r$ are constants independent of R and N .

Lemma 4. If $f(t, u)$ satisfies condition (1), $F(t, v)$ conditions (4) and (5), $R = P(|c|)^{1/n}$, $N = q(|c|)^{1/n}$, $|c|^{\varepsilon/n} > \max(\tilde{k}_1, \tilde{k}_2)$, where

$$\tilde{k}_1 = 6\tilde{B}p^{n-1}M \left\{ p + [DA(p+q) + p] \frac{1}{\pi m} \int_0^{l/2} \frac{\omega(s)}{s} ds \right\},$$

$$\tilde{k}_2 = \frac{3}{2}(1 + 2\tilde{B})Bp^{n-1}\{[p + DA(p+q)]M + rkp\},$$

p, q are numbers ≥ 1 , $m = \text{const}$, then $WS_R(N) \subset S_{R_1}(N_1)$, where $R_1 = [2M|c|]^{1/n}$, $N_1 = (rk+1) \times [2M|c|]^{1/n}$.

Lemma 5. If all the conditions of Lemma 4 are satisfied, then the operator $W\Phi^+$ is continuous on $S_R(N)$.

By virtue of the lemmas indicated above, the following is established.

Theorem. If $f(t, u)$ satisfies condition (1), and $F(t, v)$ satisfies conditions (2) and (3), or, equivalently, conditions (4) and (5), and $\chi = \text{ind } G(t) \geq 0$, then problem (*) is solvable.

Remark 1. The assumption $\chi \geq 0$ in the above arguments is essential, since the solvability of problem (*) is to a considerable extent ensured by the presence of the parameter c .

Remark 2. A problem of the form

$$\Phi^+(t) = G(t)[\Phi^-(t)]^n + F\left(t, \int_L \frac{f[\tau, \Phi^-(\tau)]}{\tau - t} d\tau\right) \quad (*')$$

by means of the change of variable $z = 1/w$ ($\tau = 1/t$) is transformed into the problem

$$[\Phi_1^+(\tau)]^n + F_1\left(\tau, \int_L \frac{f_1[\tau_1, \Phi_1^+(\tau_1)]}{\tau_1 - \tau} d\tau_1\right) = G_1(\tau)\Phi_1^-(\tau),$$

where

$$f_1[\tau_1, \Phi_1^+(\tau_1)] = f\left[\frac{1}{\tau_1}, \Phi_1^+(\tau_1)\right] \frac{\tau}{\tau_1}, \quad F_1(\tau, S\Phi_1^+) = F\left(\frac{1}{\tau}, S\Phi_1^+\right) / G\left(\frac{1}{\tau}\right).$$

Further, note that

$$\text{Ind } G_1(\tau) = \text{Ind } \frac{1}{G(1/\tau)} = \text{Ind } G(t) = \chi.$$

It is not hard to see that if $f(t, u)$ satisfies condition (1), and $F(t, v)$ satisfies conditions (2) and (3), or, equivalently, (4) and (5), then $f_1(\tau, u)$ and $F_1(\tau, v)$, respectively, satisfy analogous conditions.

Consequently, a problem of the form $(*)'$ is also solvable.

Remark 3. One can prove the solvability also of the more general problem

$$[\Phi^+(t)]^n + F \left\{ t, \Phi^+(t), \int_L \frac{f[t, \tau, \Phi^+(\tau)]}{\tau - t} d\tau \right\} = G(t) \Phi^-(t), \quad (**)$$

where the function $f(t, \tau, u)$ is defined for $t, \tau \in L$, $u = \Phi^+(z) \in E^+$, and

$$|f(t_1, \tau_1, u_1) - f(t_2, \tau_2, u_2)| \leq A\{\psi(|t_1 - t_2|) + \omega(|\tau_1 - \tau_2|) + |u_1 - u_2|\},$$

$$\omega(s), \psi(s) \in \Phi^*, \quad \psi(s) |\ln s| \leq F_0 \omega(s), \quad 0 < s \leq l, \quad F_0 = \text{const} > 0;$$

$F(t, u, v)$ is defined for $t \in L$, $u = \Phi^+(z) \in E^+$,

$$v = \int_L \frac{f(t, \tau, u)}{\tau - t} d\tau$$

and satisfies the conditions

$$|F(t, u, v)| \leq B_1(1 + |u|^{n-\varepsilon} + |v|^{n-\varepsilon}), \quad 0 < \varepsilon < n,$$

$$|F(t_1, u_1, v_1) - F(t_2, u_2, v_2)| \leq B_1\{(1 - \tilde{v}^{n-\varepsilon})\omega(|t_1 - t_2|) + (1 + \tilde{v}^{n-1-\varepsilon})[|u_1 - u_2| + |v_1 - v_2|]\},$$

$$\tilde{v} = \max(|u_i|, |v_i|), \quad i = 1, 2.$$

A problem of the form $(**)$, when the function F does not depend on the third expression, in the Hölder class $\tilde{H}_\mu(\delta)$, was considered in paper (6).

In conclusion we report that, using paper (5), it has been proved that a solution of problem $(*)$ can be obtained by the method of successive approximations.

Azerbaijan State University
named after S. M. Kirov

Received
25 IV 1966

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Note: Figure translations are in progress. See original paper for figures.

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