

GEOMETRIES OVER DEGENERATE OCTAVES

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Abstract

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MATHEMATICS

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GEOMETRIES OVER DEGENERATE OCTAVES

(Presented by Academician Yu. V. Linnik, June 7, 1966)

1. In the present note the projective plane $P_2(i, j, \varepsilon)$ over the algebra of degenerate octaves $R(i, j, \varepsilon)$ is considered, and the connection is indicated between the projective and non-Euclidean geometries of this plane and the limiting special groups of types G_2, F_4, E_6 . All considerations are carried out over the field of real numbers.
2. Let $\mathfrak{G}$ be a connected Lie group, G its Lie algebra, and

$$G = K + E \quad (1)$$

a Cartan decomposition of the Lie algebra G , i.e.

$$[K, K] \subset K; \quad [E, E] \subset K; \quad [K, E] \subset E.$$

Following F. A. Berezin and I. M. Gelfand ((¹), p. 349), we define the limiting Lie group $\mathfrak{G}_0$ for the group $\mathfrak{G}$ with respect to the Cartan decomposition (1) as follows. Let $\mathfrak{K}$ be the connected subgroup of the group $\mathfrak{G}$ corresponding to the Lie subalgebra K . If $g \in \mathfrak{K}$, then $A_{gE} \subset E$, where A_g is the image of the element g under the adjoint representation.

The group $\mathfrak{G}_0$ is constructed on the direct product of the group $\mathfrak{K}$ and the vector space E . Namely:

$$(k_1, e_1)(k_2, e_2) \stackrel{\text{def}}{=} (k_1 k_2, A_{k_2}^{-1} e_1 + e_2),$$

where $k_i \in \mathfrak{K}$, $e_i \in E$ ($i = 1, 2$).

In addition to the group $\mathfrak{G}_0$, we define the limiting Lie group with similitudes $\overline{\mathfrak{G}}_0$ for the group $\mathfrak{G}$ with respect to the same Cartan decomposition (1). We construct the group $\overline{\mathfrak{G}}_0$ on the direct product of the group $\mathfrak{G}_0$ and the set R of positive real numbers, setting

$$(k_1, e_1, \lambda_1)(k_2, e_2, \lambda_2) \stackrel{\text{def}}{=} (k_1, k_2, \lambda_2^{-1} A_{k_2}^{-1} e_1 + e_2, \lambda_1 \lambda_2),$$

where $k_i \in \mathfrak{K}$, $e_i \in E$, $\lambda_i \in R$ ($i = 1, 2$).

It is easy to verify the correctness of the definition introduced, and also the fact that the naturally embedded space E forms in $\mathfrak{G}_0$ (and in $\overline{\mathfrak{G}}_0$) a commutative normal divisor.

3. We now consider the algebra of degenerate octaves $R(i, j, \varepsilon)$. The linear space of the algebra $R(i, j, \varepsilon)$ is the direct sum of two linear spaces of the quaternion algebra $R(i, j)$, so that $\dim R(i, j, \varepsilon) = 8$. Multiplication in $R(i, j, \varepsilon)$ is defined by the formula

$$(x_1, y_1)(x_2, y_2) = (x_1 x_2, y_2 x_1 + y_1 \bar{x}_2).$$

The linear mapping

$$J : R(i, j, \varepsilon) \rightarrow R(i, j, \varepsilon), \quad (x, y) \mapsto \overline{(x, y)} = (\bar{x}, -y)$$

is an involution (i.e., an involutive antiautomorphism) in $R(i, j, \varepsilon)$.

The zero divisors (together with zero) have the form $(0, y)$ and form in $R(i, j, \varepsilon)$ an ideal E with trivial multiplication.

4. Let $\mathfrak{M}_3$ denote the space of square matrices of order 3 with entries from the algebra $R(i, j, \varepsilon)$. For $A = \|a_{ij}\| \in \mathfrak{M}_3$, put

$$A^* = \Gamma \tilde{A}^t \Gamma^{-1},$$

where

$$\tilde{A}^t = \|(\overline{a_{ij}})_{ji}\|, \quad \Gamma = \text{diag}(\gamma_1, \gamma_2, \gamma_3), \quad \gamma_i = \pm 1.$$

Put

$$\mathfrak{M}_3^+(\Gamma) = \{A : A \in \mathfrak{M}_3; A^* = A\}.$$

On the space $\mathfrak{M}_3^+(\Gamma)$ we introduce the structure of a Jordan algebra with identity $\mathfrak{B}(R(i, j, \varepsilon), \Gamma)$, putting

$$AB = \frac{1}{2}(A \cdot B + B \cdot A),$$

where $A, B \in \mathfrak{M}_3^+(\Gamma)$; $A \cdot B$ is the ordinary product of the matrices A and B . In what follows we shall consider only the cases

$$\Gamma = \Gamma_+ = \text{diag}(1, 1, 1), \quad \Gamma = \Gamma_- = \text{diag}(1, 1, -1).$$

We also define

$$\mathfrak{B}(E, \Gamma) = \{A = \|a_{ij}\| : A \in \mathfrak{B}(R(i, j, \varepsilon), \Gamma); a_{ij} \in E\}.$$

Clearly, $\mathfrak{B}(E, \Gamma)$ is an ideal in $\mathfrak{B}(R(i, j, \varepsilon), \Gamma)$ with trivial multiplication.

5. To construct the projective plane, we introduce into consideration the spaces

$$\Pi_\Gamma = \{X : X \in \mathfrak{B}(R(i, j, \varepsilon), \Gamma) \setminus \mathfrak{B}(E, \Gamma); X^2 = \text{Sp } X \cdot X\},$$

where $\text{Sp } X$ is the trace of the matrix X , and

$$\Pi_\Gamma^* = \{\rho X\},$$

where $X \in \Pi_\Gamma$, $\rho \neq 0$ is real; that is, Π_Γ^* is the set (or manifold) of lines passing through 0 and lying in Π_Γ .

We now define the projective plane $P_2(i, j, \varepsilon)$ over the algebra $R(i, j, \varepsilon)$ as the union of two sets—the set of points P and the set of lines L —with an incidence relation defined between the elements of these sets: $x \circ a$, where $x \in P$, $a \in L$, satisfying the following properties: there exist two one-to-one mappings

$$\mu_1 : P \xrightarrow{\text{onto}} \Pi_{\Gamma_+}^*, \quad \mu_2 : L \xrightarrow{\text{onto}} \Pi_{\Gamma_+}^*$$

such that, if $\rho X = \mu_1 x$, $\rho A = \mu_2 a$, then

$$x \circ a \iff XA = 0.$$

6. We now define the elliptic polar transformation (elliptic polarity) π_+ :

$$\pi_+ : P \rightarrow L, \quad x \mapsto a = \mu_2^{-1} \mu_1 x.$$

Clearly,

$$x \circ a \iff \pi_+^{-1} a \circ \pi_+ x.$$

The projective plane $P_2(i, j, \varepsilon)$ with the given polarity π_+ will be called the elliptic unitary plane $\bar{S}_2(i, j, \varepsilon)$ over the algebra $R(i, j, \varepsilon)$ (cf. (2)).

Next, we define the hyperbolic unitary plane ${}^1\bar{S}_2(i, j, \varepsilon)$. For this purpose consider the one-to-one mappings

$$\begin{aligned} \psi_1 : \Pi_{\Gamma_+}^* &\rightarrow \Pi_{\Gamma_-}^*, & \rho \begin{vmatrix} \lambda_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \lambda_2 & x_1 \\ x_2 & \bar{x}_1 & \lambda_3 \end{vmatrix} &\mapsto \rho \begin{vmatrix} \lambda_1 & x_3 & -\bar{x}_2 \\ \bar{x}_3 & \lambda_2 & -x_1 \\ x_2 & \bar{x}_1 & -\lambda_3 \end{vmatrix}; \\ \psi_2 : \Pi_{\Gamma_+}^* &\rightarrow \Pi_{\Gamma_-}^*, & \rho \begin{vmatrix} \lambda_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \lambda_2 & x_1 \\ x_2 & \bar{x}_1 & \lambda_3 \end{vmatrix} &\mapsto \rho \begin{vmatrix} \lambda_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \lambda_2 & x_1 \\ -x_2 & -\bar{x}_1 & -\lambda_3 \end{vmatrix}. \end{aligned}$$

The hyperbolic polar transformation π_- is defined as follows:

$$\pi_- : P \rightarrow L, \quad x \mapsto a = \mu_2^{-1} \psi_2^{-1} \psi_1 \mu_1 x.$$

It turns out that

$$x \circ a \iff \pi_-^{-1} a \circ \pi_- x.$$

By the plane ${}^1\bar{S}_2(i, j, \varepsilon)$ we shall mean the plane $P_2(i, j, \varepsilon)$ with the polarity π_- defined above.

In the spaces $\bar{S}_2(i, j, \varepsilon)$ and ${}^1\bar{S}_2(i, j, \varepsilon)$ one can introduce the structures of semi-Riemannian spaces (see, for example, ⁽²⁾), defining in them a metric as follows. Consider the mappings

$$p_1 : \bar{S}_2(i, j, \varepsilon) \xrightarrow{R_8} \bar{S}_2(i, j), \quad p_2 : {}^1\bar{S}_2(i, j, \varepsilon) \xrightarrow{{}^4R_8} {}^1\bar{S}_2(i, j)$$

(see ^(2, 3)).

Here by unitary non-Euclidean planes are meant the sets of interior points, R_8 is eight-dimensional Euclidean space; ${}^4R_8$ is a pseudo-Euclidean space of index 4. The mappings p_i ($i = 1, 2$) are fibrations. We shall denote the distance between points x and y by $d(x, y)$. Then

$$d(x, y) \stackrel{\text{def}}{=} d(p_i x, p_i y), \quad \text{if } p_i x \neq p_i y,$$

$$d(x, y) \stackrel{\text{def}}{=} d_R(x, y), \quad \text{if } p_i x = p_i y,$$

where $d_R(x, y)$ is the distance in the metric of the fiber.

7. A collineation φ of the plane $P_2(i, j, \varepsilon)$ will mean a pair of one-to-one mappings $\varphi = (\varphi_1, \varphi_2)$

$$\varphi_1 : P \xrightarrow{\text{onto}} P, \quad \varphi_2 : L \xrightarrow{\text{onto}} L$$

such that

$$x \circ a \iff \varphi_1 x \circ \varphi_2 a.$$

In what follows we shall write $\varphi x = \varphi_1 x$, $\varphi a = \varphi_2 a$.

A similarity of the plane $\bar{S}_2(i, j, \varepsilon)$ (or ${}^1\bar{S}_2(i, j, \varepsilon)$) is a collineation φ such that $\varphi\pi_+ = \pi_+\varphi$ (respectively, $\varphi\pi_- = \pi_-\varphi$). A motion of the planes $\bar{S}_2(i, j, \varepsilon)$, ${}^1\bar{S}_2(i, j, \varepsilon)$ is a similarity preserving the structure of semi-Riemannian spaces, i.e. the metric $d(x, y)$.

8. **Theorem 1.** The group of automorphisms of the algebra $R(i, j, \varepsilon)$ is isomorphic to the group $(G_2^+)_0$, where G_2^+ is the compact group of type G_2 .

Theorem 2. The group of collineations of the plane $P_2(i, j, \varepsilon)$ is isomorphic to the group $(\bar{E}_6^-)_0$, where E_6^- is the noncompact group of type E_6 , which is the group of collineations of the octavian projective plane $P_2(i, j, k)$ (see, for example, (3)).

Theorem 3. The groups of similarities and motions of the plane $\bar{S}_2(i, j, \varepsilon)$ are respectively isomorphic to the groups $(\bar{F}_4^+)_0$ and $(F_4^+)_0$, where F_4^+ is the compact group of type F_4 .

Theorem 4. The groups of similarities and motions of the plane ${}^1\bar{S}_2(i, j, \varepsilon)$ are respectively isomorphic to the groups $(\bar{F}_4^-)_0$ and $(F_4^-)_0$, where F_4^- is the noncompact group of type F_4 , which is the group of motions of the octavian unitary hyperbolic plane ${}^1S_2(i, j, k)$.

9. A line $P_1(i, j, \varepsilon)$ of the plane $P_2(i, j, \varepsilon)$, considered as the set of points incident with one and the same line, is a cylinder $S^4 \times R^4$ with a 4-dimensional sphere S^4 as base and a 4-dimensional plane R^4 as generator. The line $P_1(i, j, \varepsilon)$ may be regarded as the absolute in the semi-Euclidean space ${}^1S_9^5$ (see (4)). The group of motions of the space ${}^1S_9^5$ induces on the absolute the group C of conformal transformations, and the group of similarities induces the group $\bar{C}$ of conformal transformations with similarity.

On the other hand, the line $P_1(i, j, \varepsilon)$, considered as a submanifold of the semi-Riemannian space $\bar{S}_2(i, j, \varepsilon)$ (or ${}^1\bar{S}_2(i, j, \varepsilon)$), is isometric to a connected component of the sphere of the semi-Euclidean space R_9^5 (respect-

ively, ${}^{1,0}R_9^5$). Therefore, after identifying diametrically opposite points, the lines of the planes $\bar{S}_2(i, j, \varepsilon)$, ${}^1\bar{S}_2(i, j, \varepsilon)$ will be isometric respectively to the spaces S_8^4 and ${}^{1,0}S_8^4$ (see (4)). Moreover, the following theorems hold:

Theorem 5. The group of collineations $(\bar{E}_6^-)_0$ of the plane $P_2(i, j, \varepsilon)$ induces on the line $P_1(i, j, \varepsilon)$ the group $\bar{C}$, while the subgroup $(E_6^-)_0$ induces the group C (at least locally).

Theorem 6. The group of similarities of the plane $\bar{S}_2(i, j, \varepsilon)$ (respectively, ${}^1\bar{S}_2(i, j, \varepsilon)$) induces on the line, as on the space S_8^4 (respectively, on the space ${}^{1,0}S_8^4$), the group of similarities, and the group of motions—the group of motions (at least locally).

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