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Abstract

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MATHEMATICS

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REGULARIZATION OF ILL-POSED PROBLEMS FOR SINGULAR INTEGRAL EQUATIONS

(Presented by Academician A. N. Tikhonov, 19 XII 1966)

Numerous works on singular integral equations have been devoted to the construction of a regularizer that would reduce a singular integral equation to an equivalent Fredholm equation ⁽¹⁻⁷⁾. At the same time, for a general closed linear operator A in Hilbert space, a necessary condition for the existence of a regularizer is the closedness of the range $R(A)$ ^(5,7). This is equivalent to the well-posedness of the problem $Au = f$, provided that u is orthogonal to the subspace of solutions of $Av = 0$.

Therefore, for an ill-posed integral equation with a singular kernel we shall pose a different regularization problem—regularization in the sense of A. N. Tikhonov ^(8,9), i.e., we shall abandon the principal requirement: the equivalence of the regularized and the original problems. The discussion will concern only the uniform convergence of solutions of a family of regularized equations to some solution of the original problem. We shall prove that a certain modification of A. N. Tikhonov's regularization, developed by him for bounded operators, is also applicable to unbounded operators, i.e., to integral equations with singular kernels, and also to ill-posed linear integro-differential equations with partial derivatives.

Let H be a Hilbert space, and let T be a closed operator in H with dense domain $D(T)$. Consider the equation

$$Tv = f, \tag{1}$$

where $v \in D(T)$, $f \in R(T)$ is given.

In the case of bounded operators, the regularization proposed for this problem by A. N. Tikhonov has the form

$$(T^*T + \delta)x_\delta = T^*f_\delta; \quad \delta \leq \|f_\delta - f\|. \tag{2}$$

We shall suppose that the operator T is unbounded. Assume first that $f_\delta \in D(T^*)$.

Denote by P_T, P_{T^*} the projection operators onto the subspaces of solutions of $Tu = 0$ and $T^*u = 0$, respectively; by T_1 the restriction of the operator T to the domain $(1 - P_T)D(T)$; and by T_{1*} the restriction of the operator T^* to the domain $(1 - P_{T^*})D(T^*)$. It is obvious that: 1) T_1^{-1}, T_{1*}^{-1} exist; 2) $TT_1^{-1} = 1, \overline{T_1^{-1}T} = 1 - P_T$; 3) $\overline{P_T T_1^{-1}} = 0; \overline{P_{T^*} T_{1*}^{-1}} = 0$; 4) $\overline{P_{T^*} T} = 0, \overline{P_T T^*} = 0$. (The bar denotes closure.) The operator $A = T_1^{-1}(1 - P_{T^*})$ is defined on an everywhere dense set in H , and therefore the operators A^* and A^*A , whose domains are dense, exist. It is not difficult to verify that $A^* = T_{1*}^{-1}(1 - P_T)$. We shall prove this assertion.

* By P_{TH} we mean the subspace of solutions of $Tx = 0$; $(1 - P_T)H$ is its orthogonal complement. By $(1 - P_T)D$, where $D \subset H$, we mean the domain equal to $D \cap (1 - P_T)H$.

Proof. Let $f \in D[T_1^{-1}(1 - P_T)] = R(T^*) \oplus P_TH$, and let $F = T_{1*}^{-1}(1 - P_T)f$; then $P_{T^*}F = 0$ and $(1 - P_T)f = T^*F$, while

$$\begin{aligned} (f, Ag) &= (f, T_1^{-1}(1 - P_{T^*})g) = (f(1 - P_T)T_1^{-1}(1 - P_{T^*})g) = ([1 - P_T]f, T_1^{-1}[1 - P_{T^*}]g) \\ &= (T^*F, T_{1*}^{-1}[1 - P_{T^*}]g) = (F, [1 - P_{T^*}]g) = (F, g), \end{aligned}$$

i.e. $A^*f = F \Rightarrow T_1^{-1}(1 - P_T) \subset A^*$. Let $g \in D(T)$, $h \in D(A^*)$, $A^*h = f$; then

$$\begin{aligned} (f, Tg) &= (h, ATg) = ([1 - P_T]h, g) \Rightarrow T^*f = (1 - P_T)h \Rightarrow (1 - P_T)D(A^*) = R(T^*) \Rightarrow D(A^*) = R(T^*) \oplus P_TH \\ &\Rightarrow A^* = T_{1*}^{-1}(1 - P_T). \end{aligned}$$

As indicated above, the operator

$$AA^* = T_1^{-1}(1 - P_{T^*})T_{1*}^{-1}(1 - P_T) = T_1^{-1}T_{1*}^{-1}(1 - P_T)$$

has an everywhere dense domain of definition. Consequently, the operator $T_1^{-1}T_{1*}^{-1}$ is defined on a domain everywhere dense in $(1 - P_T)H$.

By the known von Neumann lemma, for any closed operator B with an everywhere dense domain of definition, the operators $(1 + B^*B)^{-1}$ and $B(1 + B^*B)^{-1}$ are defined on all of H and are bounded by one. Consequently, $(B[1 + B^*B])^* = (1 + B^*B)B^*$ is bounded by one.

Putting $B = \delta^{-1/2}T$, we arrive at the conclusion that the operators

$$C_\delta = (\delta + T^*T)^{-1}$$

and

$$B_\delta = (\delta + T^*T)^{-1}T^*$$

are bounded respectively by δ^{-1} and $\delta^{-1/2}$.

It is also obvious that the operator

$$R_\delta = (\delta + T^*T)^{-1}T^*T_1$$

is defined on $(1 - P_T)H$ and is bounded by two. Indeed,

$$\begin{aligned}\|R_\delta\| &= \|(\delta + T^*T)^{-1}T^*T_1\| \leq \|(\delta + T^*T)^{-1}T^*T\| \\ &= \|1 - \delta(\delta + T^*T)^{-1}\| \leq 2.\end{aligned}$$

Since $P_{T^*}T_1 = P_{T^*}T = 0$, and hence $T^*T_1 = T_{1^*}T_1$, we have

$$R_\delta = (\delta + T^*T)^{-1}T_{1^*}T_1.$$

It follows from this that

$$R_\delta^{-1} = T_1^{-1}T_{1^*}^{-1}(\delta + T^*T) = \delta T_1^{-1}T_{1^*}^{-1} + 1 - P_T$$

exists and is defined on a domain dense in $(1 - P_T)H$. On elements of this domain, obviously, $R_\delta^{-1} \rightarrow 1$ as $\delta \rightarrow 0$.

Let $z = (1 - P_T)v$.

Since $\|R_\delta\| \leq 2$, by Theorem 3.2 of Part I of the book ⁽¹⁰⁾, we obtain that R_δ converge as $\delta \rightarrow 0$ to 1 on the subspace $(1 - P_T)H$. Consequently,

$$\begin{aligned}x_\delta - z &= B_\delta f_\delta - z = B_\delta(f_\delta - f) + B_\delta f - z \\ &= B_\delta(f_\delta - f) + B_\delta T_1 z - z = B_\delta(f_\delta - f) + R_\delta z - z \rightarrow 0 \quad \text{as } \delta \rightarrow 0,\end{aligned}$$

since

$$\|B_\delta(f_\delta - f)\| \leq \frac{1}{\sqrt{\delta}}\|f_\delta - f\| = \sqrt{\delta} \rightarrow 0,$$

and $z \in (1 - P_T)H$, and hence $\|R_\delta z - z\| \rightarrow 0$ as $\delta \rightarrow 0$.

In solving the well-posed problem (2), we proceeded from the fact that $f_\delta \in D(T^*)$. This restriction can be avoided by means of the following procedure.

Consider the equation

$$(T^*T + \delta_1)y_\delta = T^*u_\delta, \quad \delta_1 = \|(1 + \delta TT^*)^{-1} - 1\|f_\delta\|, \quad (3)$$

where u_δ , in turn, satisfies the equation

$$(1 + \delta TT^*)u_\delta = f_\delta, \quad f_\delta \in H; \quad \|f_\delta - f\| \leq \delta, \quad f \in R(T), \quad (4)$$

f_δ, f are given.

In this case it is obvious that T^*u_δ exists for any $f_\delta \in H$. Moreover, $\delta_1 \rightarrow 0$ as $\delta \rightarrow 0$, and

$$\|u_\delta - f\| \leq \delta + \delta_1 \rightarrow 0 \quad \text{as } \delta \rightarrow 0.$$

Indeed,

$$\delta_1 = \|[(1 + \delta TT^*)^{-1} - 1](f_\delta - f + f)\| \leq 2\delta + \|[(1 + \delta TT^*)^{-1} - 1]f\| \rightarrow 0 \quad \text{as } \delta \rightarrow 0,$$

and

$$\|u_\delta - f\| = \|(1 + \delta TT^*)^{-1} f_\delta - f_\delta + (f_\delta - f)\| \leq \|[(1 + \delta TT^*)^{-1} - 1]f_\delta\| + \delta = \delta_1 + \delta.$$

Thus, in view of the preceding considerations, $y_\delta - z \rightarrow 0$ as $\delta \rightarrow 0$.

Suppose now that $H = L_2[R^n]$. Consider

$$v_{\delta_1, \varepsilon} = \varepsilon^{-(n+2)/2} (R_{\delta_1} - 1)(1 - P_T) e^{-(x-\xi)^2/4\varepsilon}, \quad x, \xi \in R^n,$$

i.e., $v_{\delta_1, \varepsilon}$ satisfies the equation

$$(\delta_1 + T^*T)v_{\delta_1, \varepsilon} = \delta_1(1 - P_T) e^{-(x-\xi)^2/4\varepsilon} \varepsilon^{-(n+2)/2}.$$

By what has been proved, for fixed ε the function $v_{\delta_1, \varepsilon}$ tends to zero in norm as $\delta_1 \rightarrow 0$. Consequently, there exists a function $\varepsilon = \varepsilon(\delta_1) \rightarrow 0$ as $\delta_1 \rightarrow 0$ such that $\varepsilon(\delta_1) > \delta_1^{2/(4+n)}$ and $\|v_{\delta_1, \varepsilon(\delta_1)}\|$ is bounded above as $\delta_1 \rightarrow 0$.

Let us prove that if $z(x) \in (1 - P_T)L_2(R^n) \cap C^2$ and $Tz(x) = f$, then

$$\max_x \left| z(x) - \frac{1}{(2\pi\varepsilon(\delta_1))^{n/2}} \int y_\delta(\xi) e^{-(x-\xi)^2/4\varepsilon(\delta_1)} d\xi \right| = O[\varepsilon(\delta_1)]_{\delta \rightarrow 0}. \quad (5)$$

Indeed,

$$\begin{aligned} z(x) - \frac{1}{(2\pi\varepsilon)^{n/2}} \int e^{-(x-\xi)^2/4\varepsilon} y_\delta(\xi) d\xi \\ &= \frac{1}{(2\pi\varepsilon)^{n/2}} \int e^{-(x-\xi)^2/4\varepsilon} [z(x) - z(\xi)] d\xi + \frac{1}{(2\pi\varepsilon)^{n/2}} \int e^{-(x-\xi)^2/4\varepsilon} [z(\xi) - y_\delta(\xi)] d\xi \\ &\leq \varepsilon \max_{x_i x_j} z_{x_i x_j} + \left| \frac{1}{(2\pi\varepsilon)^{n/2}} (B_{\delta_1}[y_\delta - f] + B_{\delta_1}f - z, e^{-(x-\xi)^2/4\varepsilon}) \right| \\ &\leq \varepsilon \max_{x_i x_j} z_{x_i x_j} + \frac{\delta_1 + \delta}{\sqrt{\delta_1}(2\pi\varepsilon)^{n/2}} \|e^{-(x-\xi)^2/4\varepsilon}\| + \frac{1}{(2\pi\varepsilon)^{n/2}} \left| ((R_{\delta_1} - 1)(1 - P_T)z, e^{-(x-\xi)^2/4\varepsilon}) \right| \\ &\leq \varepsilon \max_{x_i x_j} z_{x_i x_j} + \frac{(\delta_1 + \delta)(2\pi\varepsilon)^{-n/4}}{\sqrt{\delta_1}} + \|z\| \varepsilon \|v_{\delta_1, \varepsilon}\| = O(\varepsilon). \end{aligned}$$

Thus we have proved the following theorem:

Theorem. For any closed linear operator T with everywhere dense domain of definition in $L_2[R^n]$, problem (3)–(4) is well posed, and its solution y_δ satisfies

relation (5), if there exists a solution of the equation $Tz(x) = f$ belonging to $C^q \cap (1 - P_T)D(T)$.

We note that under the additional condition of Tikhonov well-posedness of problem (1), using the effective technique of M. M. Lavrent'ev (11), one can obtain refined estimates for the regularization presented here.

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