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**Abstract**

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**MATHEMATICS**

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# A HARNACK INEQUALITY FOR GENERALIZED SOLUTIONS OF QUASILINEAR PARABOLIC EQUATIONS OF SECOND ORDER

*(Presented by Academician V. I. Smirnov on 30 V 1966)*

In the works <sup>(1,2)</sup> J. Moser proposed a method for obtaining the Harnack inequality for generalized solutions (g.s.) from the classes  $\dot{W}_2^1(\Omega)$  and  $\dot{W}_2^1(Q)$  of the equations  $(\mathcal{A}_{ij}u_{x_i})_{x_j} = 0$  and  $u_t = (\mathcal{A}_{ij}u_{x_i})_{x_j}$ . In <sup>(3)</sup> J. Serrin extended the result of <sup>(1)</sup> to g.s. of a certain class of quasilinear elliptic equations. Here we extend the Harnack inequality to g.s. of a certain class of quasilinear parabolic equations of second order (see <sup>(8)</sup>), and in the proof the approaches used in <sup>(1-3)\*</sup> are developed.

Let  $\Omega$  be some bounded  $n$ -dimensional domain in  $E_n$ ,  $Q$  the cylinder  $\Omega \times [t_1, t_2]$ . By  $L_{p,p_1}(Q)$  ( $p, p_1 \geq 1$ ) we denote the space of all measurable functions in  $Q$  for which the norm is finite

$$\|u\|_{p,p_1,Q} = \left( \int_{t_1}^{t_2} \|u\|_{p,\Omega}^{p_1} dt \right)^{1/p_1} = \left( \int_{t_1}^{t_2} \left( \int_{\Omega} |u|^p dx \right)^{p_1/p} dt \right)^{1/p_1}. \quad (1)$$

For  $p_1 = \infty$ ,

$$\|u\|_{p,\infty,Q} = \text{vrai max}_{t \in [t_1, t_2]} \|u\|_{p,\Omega}.$$

We introduce also the following notation.  $\dot{W}_2^{1,0}(Q)$  is the closure of the set of all smooth and finite in  $Q$  functions in the norm  $(\|u\|_{2,Q}^2 + \|u_x\|_{2,Q}^2)^{1/2}$ ;  $\dot{W}_2^{1,1}(Q)$  is the subset of all functions from  $\dot{W}_2^{1,0}(Q)$  which vanish in some neighborhood of the lateral surface and the lower base of the cylinder  $Q$  and have the generalized derivative  $u_t \in L_2(Q)$ ;  $U_2^{1,0}(Q)$  is the space of all functions  $u(x, t)$  belonging to  $L_2(Q)$  and having generalized derivatives  $u_{x_i}$  from  $L_2(Q)$  and a finite  $\text{vrai max}_{t \in [t_1, t_2]} \|u\|_{2,\Omega}$ , with norm

$$\langle\langle u \rangle\rangle_Q = \left( \text{vrai max}_{t \in [t_1, t_2]} \|u\|_{2, \Omega}^2 + \|u_x\|_{2, Q}^2 \right)^{1/2}, \quad (2)$$

where

$$u_x = (u_{x_1}, \dots, u_{x_n}), \quad \|u_x\|_{2, Q} = \left\| \left( \sum_{i=1}^n u_{x_i}^2 \right)^{1/2} \right\|_{2, Q}.$$

Denote by  $K_\rho(x_0)$  the  $n$ -dimensional ball of radius  $\rho$  ( $\rho > 0$ ) with center at the point  $x_0$ , and by  $Q_\rho(x_0, t_0)$  the cylinder  $K_\rho(x_0) \times [t_0 - \rho^2, t_0]$ .

**Lemma 1.** Let  $u(x, t) \in U_2^{1,0}(Q_\rho)$ , and let the numbers  $l, l_1$  ( $l, l_1 \geq 1$ ) satisfy the conditions

$$1/l + 2/nl_1 \geq 1/2,$$

$$l \geq 2; \quad l_1 \geq 2 \text{ for } n \geq 3; \quad 2 \leq l < \infty, \quad l_1 > 2 \text{ for } n = 2; \quad l \geq 2, \quad l_1 \geq 4 \quad \text{for } n = 1. \quad (3)$$

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\* A certain generalization of the result of Moser's work <sup>(2)</sup> was obtained in <sup>(4)</sup>, where it was shown that the Harnack inequality will also hold for a somewhat broader class of g.s. of the equation  $u_t = (\mathcal{A}_{ij} u_{x_i})_{x_j}$  than  $\dot{W}_2^1(Q)$ .

Then the function  $u(x, t)$  belongs to the space  $L_{l, l_1}(Q_\rho)$  and satisfies the inequality

$$\|u\|_{l, l_1, Q_\rho} \leq a_0 \left( \text{vrai max}_{t \in [t_1, t_2]} \|u\|_{2, K_\rho} \right)^{1-2/l_1} \left( \|u\|_{2, Q_\rho} + \|u_x\|_{2, Q_\rho} \right)^\kappa \left( \|u\|_{2, Q_\rho} \right)^{2/l-\kappa}, \quad (4)$$

where  $\kappa = n/2 - n/l$  (obviously,  $\kappa \in [0, 2/l_1]$ ) and  $a_0$  depends only on  $n$  for  $n \neq 2$  and on  $l$  for  $n = 2$ .

In the case  $1/l + 2/nl_1 > 1/2$ , inequality (4) implies the inequality

$$\|u\|_{l, l_1, Q_\rho} \leq \varepsilon \langle u \rangle_{Q_\rho} + a'_0 \varepsilon^{-\frac{1}{\kappa-2/l_1} \frac{1+\kappa-2/l_1}{\kappa-2/l_1}} \|u\|_{2, Q_\rho}, \quad (5)$$

where  $\varepsilon$  is an arbitrary positive number.

Consider the equation

$$\frac{\partial u}{\partial t} - \frac{\partial}{\partial x_i} \mathcal{L}_i(x, t, u, u_x) + \mathcal{L}_0(x, t, u, u_x) = 0, \quad (6)$$

where  $\mathcal{L}_i(x, t, u, p)$  and  $\mathcal{L}_0(x, t, u, p)$  ( $p = (p_1, \dots, p_n)$ ) are given functions of the variables  $x, t, u, p$ , defined in the domain

$$\mathcal{R} : x \in \Omega, t \in [0, T], -\infty < u < \infty, -\infty < p_k < \infty, k = 1, \dots, n,$$

measurable in  $x, t$  in the cylinder  $Q_T = \Omega \times [0, T]$  for fixed  $u$  and  $p$ , and continuous in the variables  $u$  and  $p$  for fixed  $x$  and  $t$ . Suppose that for all  $(x, t, u, p) \in \mathcal{R}$  the inequalities

$$\left[ \sum_{i=1}^n \mathcal{L}_i^2(x, t, u, p) \right]^{1/2} \leq \mu |p| + \mathcal{A} |u| + \mathcal{F},$$

$$|\mathcal{L}_0(x, t, u, p)| \leq \mathcal{B} |p| + \mathcal{C} |u| + \mathcal{G}, \quad (7)$$

$$\mathcal{L}_i(x, t, u, p) p_i \geq \nu |p|^2 - \mathcal{D} |u|^2 - \mathcal{H},$$

hold, where  $\nu$  and  $\mu$  are positive constants;  $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{F}, \mathcal{G}, \mathcal{H}$  are nonnegative functions of the variables  $x, t$ , belonging to the spaces

$$\mathcal{A}, \mathcal{B}, \mathcal{F} \in L_{q, \infty}(Q_T), \quad \mathcal{C}, \mathcal{D}, \mathcal{G}, \mathcal{H} \in L_{q/2, \infty}(Q_T), \quad q > n \geq 2. \quad (8)$$

A generalized solution from the class  $\mathring{W}_2^{1,0}(Q_T)$  of equation (6), whose coefficients satisfy conditions (7), (8), is a function

$$u(x, t) \in U_2^{1,0}(Q_T),$$

satisfying, for almost all  $t_1, t_2 \in [0, T]$  and all

$$\Phi \in \mathring{W}_2^{1,1}(Q), \quad Q = \Omega \times [t_1, t_2],$$

the integral identity

$$\int_{\Omega} u \Phi dx \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_{\Omega} [-u \Phi_t + \mathcal{L}_i(x, t, u, u_x) \Phi_{x_i} + \mathcal{L}_0(x, t, u, u_x) \Phi] dx dt = 0. \quad (9)$$

Denote by  $\Gamma_T$  the set of all points  $(x, t)$  lying on the lateral surface or on the lower base of the cylinder  $Q_T$ , and let

$$\widehat{Q}_{2r} = \widehat{Q}_{2r}(x_0, t_0) = Q_{2r}(x_0, t_0 + 2r^2) = K_{2r}(x_0) \times [t_0 - 2r^2, t_0 + 2r^2].$$

With the aid of the main lemma of [2] and Lemma 1 one proves

**Theorem 1.** Let  $u(x, t)$  be a generalized solution from the class  $U_2^{1,0}(Q_T)$  of equation (6), whose coefficients satisfy conditions (7), (8). Suppose that this generalized solution is nonnegative in the cylinder  $\widehat{Q}_{2r}(x_0, t_0)$ , contained in  $Q_T$  and at a positive distance from  $\Gamma_T$ . Then the function  $u(x, t)$  satisfies the inequality (Harnack)

$$\text{vrai max}_{Q_{r/4}^-} u \leq \text{const} \left( \text{vrai min}_{Q_{r/4}^+} u + k_r \right), \quad (10)$$

where

$$Q_{r/4}^- = Q_{r/4}(x_0, t_0 - \frac{7}{4}r^2), \quad Q_{r/4}^+ = Q_{r/4}(x_0, t_0 + 2r^2),$$

$$k_r = r^\delta \|\mathcal{F}\|_{q,\infty,\widehat{Q}_{2r}} + r^{2\delta} \|\mathcal{G}\|_{q/2,\infty,\widehat{Q}_{2r}} + r^\delta \left( \|\mathcal{H}\|_{q/2,\infty,\widehat{Q}_{2r}} \right)^{1/2}, \quad \delta = 1 - \frac{n}{q}.$$

The constant in (10) depends only on

$$n, \nu, \mu, q, r^\delta \|\mathcal{A}, \mathcal{B}\|_{q,\infty,\widehat{Q}_{2r}}$$

and

$$r^{2\delta} \|\mathcal{C}, \mathcal{D}\|_{q/2,\infty,\widehat{Q}_{2r}}.$$

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\* By  $\frac{\partial}{\partial x_i} \mathcal{L}_i(x, t, u, u_x)$  is meant the total derivative of the function  $\mathcal{L}_i(x, t, u(x, t), u_x(x, t))$  with respect to the variable  $x_i$ .

As a consequence of Harnack's inequality one easily obtains an estimate of the Hölder constant of an arbitrary generalized solution from the class  $\mathcal{V}_2^{1,0}(Q_T)$  of equation (6), considered under conditions (7), (8).

In the case

$$\mathcal{L}_i(x, t, u, p) = \sum_{j=1}^n L_{ij} p_j + A_i u + F_i, \quad \mathcal{L}_0(x, t, u, p) = \sum_{j=1}^n B_j p_j +$$

$$+Cu + G$$

the estimate of the Hölder constant was first obtained by different methods by O. A. Ladyzhenskaya and N. N. Ural'tseva in <sup>(6)</sup>, and (for  $A_i = B_i = F_i = C = G = 0$ ) by J. Nash in <sup>(5)</sup>.

**Theorem 2.** Let  $u(x, t)$  be an arbitrary generalized solution from the class  $\mathcal{V}_2^{1,0}(Q_T)$  of equation (6), satisfying conditions (7), (8), and let  $Q' = \Omega' \times [\delta', T]$ ,  $Q'' = \Omega'' \times [\delta'', T]$ ,  $\bar{\Omega}' \subset \Omega'' \subset \bar{\Omega}'' \subset \Omega$ ,  $\delta' > \delta'' > 0$ . Then the function  $u(x, t)$  has a finite  $\text{vrai max}_{Q'} |u|$  and, for almost all  $(x, t)$ ,  $(\bar{x}, \bar{t}) \in Q'$ , satisfies the inequality

$$|u(x, t) - u(\bar{x}, \bar{t})| \leq \text{const} (\text{vrai max}_{Q'} |u| + K) (|x - \bar{x}|^2 + |t - \bar{t}|)^\gamma, \quad (11)$$

where  $\gamma \in (0, 1]$ ,

$$K = \|\mathcal{F}\|_{q,\infty,Q''} + \|\mathcal{G}\|_{q/2,\infty,Q''} + (\|\mathcal{H}\|_{q/2,\infty,Q''})^{1/2},$$

and the constant in (11) depends only on  $n, \nu, \mu, q$ , the norms  $\|\mathcal{A}, \mathcal{B}\|_{q,\infty,Q''}$ ,  $\|\mathcal{C}, \mathcal{D}\|_{q/2,\infty,Q''}$ , on the geometry of  $Q'$ , and on the distance from  $Q'$  to the lateral surface and the lower base of the cylinder  $Q''$ .

The functions  $\mathcal{A}, \mathcal{B}, \dots, \mathcal{H}$  may be regarded as elements of spaces  $L_{p,p_1}(Q_T)$  of general form ( $p, p_1 \geq 1$ ). In this case, for generalized solutions of the quasilinear equation (6) satisfying conditions (7), theorems (see <sup>(8)</sup>) analogous to Theorems 1 and 3-7 of paper <sup>(7a)</sup>, concerning linear parabolic equations, are proved.

*Remark added in proof.*

1. The results of Theorems 1 and 2 remain valid if the functions  $\mathcal{A}, \mathcal{B}, \dots, \mathcal{H}$  are characterized in terms of belonging to spaces  $L_{p,p_1}(Q_T)$  of general form, i.e., if instead of (8) one imposes the condition

$$\mathcal{A}, \mathcal{B}, \mathcal{F} \in L_{2p,2p_1}(Q_T), \quad \mathcal{C}, \mathcal{D}, \mathcal{G}, \mathcal{H} \in L_{p,p_1}(Q_T)$$

$$n/p + 2/p_1 < 2, \quad p \geq 1, \quad p_1 \geq 1, \quad n \geq 1. \quad (8')$$

2. In the case of bounded generalized solutions the results of these theorems remain valid if, instead of (7), one assumes that

$$\left[ \sum_{i=1}^n \mathcal{L}_i^2 \right]^{1/2} \leq \mu |p| + \mathcal{F}, \quad |\mathcal{L}_0| \leq \mu_1 |p|^2 + \mathcal{G}, \quad \mathcal{L}_i p_i \geq \nu |p|^2 - \mathcal{H}, \quad (7')$$

where  $\mathcal{F}, \mathcal{G}$ , and  $\mathcal{H}$  satisfy (8').

3. During the International Congress of Mathematicians in Moscow in 1966 it became known to us that, recently, a result analogous to Theorem 1 in the case of conditions (7), (8<sup>1</sup>) had been obtained by D. G. Aronson and J. Serrin (an announcement of their result appears in *Notices of the*

*Am. Math. Soc.*, 13 April 1966, p. 381), and, in the case of bounded generalized solutions under conditions (7<sup>1</sup>), (8'), also by N. S. Trudinger. In addition, for linear equations Harnack's inequality has been extended by L. P. Kuptsov to a certain class of non-hyperbolic equations. These results have not yet been published.

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*Note: Figure translations are in progress. See original paper for figures.*

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