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Abstract

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MATHEMATICS

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THE CATEGORY OF HARISH-CHANDRA MODULES OVER THE LIE ALGEBRA OF THE LORENTZ GROUP

In essence, this work investigates irreducible infinitesimal representations of the Lorentz group. The main result of the article is Theorem 4, which shows that in a special case a series of new, completely nontrivial invariants of representations arises. The authors have succeeded in reducing the problem to a purely algebraic one by introducing a certain special class of infinite-dimensional modules, which we have called Harish-Chandra modules.

Let L be a semisimple Lie algebra over the field of real numbers. Denote by L_k the subalgebra corresponding to some maximal compact subgroup. Consider a module M over the Lie algebra.

Definition. A module M is called a **Harish-Chandra module** over the Lie algebra L if, when regarded as a module over L_k , it is a direct sum $\oplus_i M_i$ of submodules M_i . Here M_i is an irreducible module over L_k , and for each M_i in the sequence $\{M_i\}$ there exist no more than finitely many equivalent to it.

A Harish-Chandra module is called irreducible if it cannot be decomposed into a direct sum of irreducible modules.

1. In this work the irreducible Harish-Chandra modules over the Lie algebra L of the proper Lorentz group are classified. The maximal compact subgroup in this case is isomorphic to the group of rotations of three-dimensional space. By L_k we shall denote the corresponding subalgebra. (In what follows, the notation L and L_k has only this meaning.) In the Lie algebra L one can choose a basis $(h_+, h_-, h_3, f_+, f_-, f_3)$ of six elements. Here the elements h_+, h_-, h_3 form a basis of the subalgebra L_k . A representation in the space M is determined by the images of the basis elements of the algebra L . They will be denoted by $H_+, H_-, H_3, F_+, F_-, F_3$. Recall that these operators H and F are related by the relations

$$\begin{aligned}
[H_+, H_3] &= -H_+; & [H_-, H_3] &= H_-; & [H_+, H_-] &= 2H_3; \\
[F_+, H_+] &= [F_-, H_-] = [F_3, H_3] = 0; \\
[H_+, F_3] &= [F_+, H_3] = -F_+; & [H_-, F_3] &= [F_-, H_3] = F_-; \\
[H_+, F_-] &= [F_-, H_+] = 2F_3; & [F_+, F_3] &= H_+; & [F_-, F_3] &= -H_-; \\
[F_+, F_-] &= -2H_3.
\end{aligned} \tag{1}$$

Let further $U(L)$ be the enveloping algebra of the Lie algebra L ; let Z be the center of the enveloping algebra. In the center Z one can choose two generators. The operators corresponding to these generators will be denoted by Δ_1 and Δ_2 . They are called Laplace operators and are expressed as follows in terms of the operators H and F :

$$\Delta_1 = \frac{1}{2}(H_-F_+ + F_-H_+) + H_3F_3; \quad \Delta_2 = H_-H_+ - F_-F_+ + H_3^2 - F_3^2 + 2H_3. \tag{2}$$

Theorem 1. *A Harish-Chandra module over the Lie algebra of the proper Lorentz group is decomposable into a direct sum of a countable number of irreducible modules. In each irreducible module the Laplace operators Δ_1 and Δ_2 have one eigenvalue each, λ_1 and λ_2 , respectively.*

We say that an operator A has one eigenvalue λ in a Harish-Chandra module if there exists such a number k that $(A - \lambda E)^k = 0$.

Assertion 1. Let M and M' be two Harish-Chandra modules over the Lie algebra L , and suppose that in each of them the Laplace operators have one eigenvalue each, λ_1, λ_2 and λ'_1, λ'_2 , respectively. $\text{Hom}(M, M') \neq 0$ if and only if $\lambda_1 = \lambda'_1$ and $\lambda_2 = \lambda'_2$.

Thus, the study of the category of Harish-Chandra modules over the algebra L may be reduced to the study of a category of modules in each of which the Laplace operators have one eigenvalue each. We shall denote this category by $C(\lambda_1, \lambda_2)$.

Assertion 2. Let $M \in C(\lambda_1, \lambda_2)$, and let M be an indecomposable module. Then there exists a nonnegative number l_0 , integral or half-integral, and the numbers $\lambda_1, \lambda_2, l_0$ are related by

$$l_0^4 + (1 + \lambda_2)l_0^2 - \lambda_1^2 = 0. \tag{3}$$

Let l_1 be a number satisfying the identities

$$l_1^2 l_0^2 = -\lambda_1^2, \quad l_1^2 + l_0^2 = -1 - \lambda_2. \tag{4}$$

We shall call the pair of eigenvalues (λ_1, λ_2) special if l_1 is real and the difference $(|l_1| - l_0)$ is a positive integer. The category $C(\lambda_1, \lambda_2)$ with such a pair will be

called special. In the opposite case the pair and the category $C(\lambda_1, \lambda_2)$ will be called nonspecial.

2. The nonspecial case. Introduce the category S . Any object of it is a pair (P, a) , where P is a finite-dimensional vector space and a is a linear nilpotent mapping $a : P \rightarrow P$. The morphisms $\gamma : (P_1, a_1) \rightarrow (P_2, a_2)$ are all such linear mappings $\gamma : P_1 \rightarrow P_2$ that $a_2\gamma = \gamma a_1$.

Theorem 2. The nonspecial category $C(\lambda_1, \lambda_2)$ is equivalent to the category S .

Thus, indecomposable modules M correspond to indecomposable objects in the category S . An indecomposable object in S is such a pair (P, a) in which the transformation a is represented by a matrix with one Jordan cell. Consequently, an indecomposable nonspecial module is determined by the following invariants: the numbers λ_1 and λ_2 , and the integer parameter n , the size of the Jordan cell.

3. The special category $C(\lambda_1, \lambda_2)$. Introduce the category S_0 . Any object A of it consists of two arbitrary finite-dimensional spaces P_1 and P_2 over the field F , and any three mappings $d_+ : P_1 \rightarrow P_2$, $d_- : P_2 \rightarrow P_1$, $\delta : P_2 \rightarrow P_2$, which satisfy the condition $d_-\delta = \delta d_+ = 0$ and the requirement that the operators δ and d_+d_- be nilpotent. The morphisms $\Gamma : A_1 \rightarrow A_2$ are all such collections of linear mappings (γ_1, γ_2) for which the diagram is commutative

$$\begin{array}{ccccccc} P_{1,1} & \xrightarrow{d_+} & P_{1,2} & \xrightarrow{\delta} & P_{1,2} & \xrightarrow{d_-} & P_{1,1} \\ \gamma_1 \downarrow & & \gamma_2 \downarrow & & \gamma_2 \downarrow & & \gamma_1 \downarrow \\ P_{2,1} & \xrightarrow{d_+} & P_{2,2} & \xrightarrow{\delta} & P_{2,2} & \xrightarrow{d_-} & P_{2,1} \end{array} \quad (5)$$

In what follows we shall assume that the field F is the field of complex numbers; however, the assertion about the category S_0 will not change if F is an arbitrary algebraically closed field of characteristic not equal to zero.

Theorem 3. The special category $C(\lambda_1, \lambda_2)$ is isomorphic to the category S_0 .

It follows from Theorem 3 that the indecomposable modules in the special category $C(\lambda_1, \lambda_2)$ are put in one-to-one correspondence with the indecomposable objects of the category S_0 , and conversely. Let us give a description of the canonical form of an indecomposable object into a direct sum in the category S_0 . Indecomposable objects may be of two types. Objects of the first type are called nonclosed, and of the second—closed. The simplest type of a nonclosed object will be called a ce-

point. If the object A is a chain, then in the space P_1 one can choose a basis of n vectors $e_1, e_2, \dots, e_{n-1}, e_n$, and in the space P_2 a basis of $(n + m + 1)$ vectors $f_0, f_1, \dots, f_n; f'_1, f'_2, \dots, f'_m$. The operators d_+, d_-, δ are then defined as follows:

$$d_- f_i = e_{i+1} \quad (i < n); \quad d_- f_n = 0; \quad d_- f'_i = 0; \quad d_+ e_i = f_i;$$

$$\delta f_i = 0 \ (i > 0); \quad \delta f_0 = f'_1; \quad \delta f'_i = f'_{i+1} \ (i < m); \quad \delta f'_m = 0.$$

The vectors f_n and f'_m in the chain will be called tail vectors. The invariants of a chain are two numbers (n, m) .

An unclosed object is composed of chains and is specified by a set of numbers

$$(s, n_1, m_1; n_2, m_2; n_3, m_3; \dots, n_k, m_k),$$

where s is equal either to 0 or to 1, and a pair of numbers n_i, m_i determines the i -th chain, with $n_1 \geq 0$; $n_i > 0 \ (i \neq 1)$; $m_i > 0 \ (i \neq k)$; $m_k \geq -1$. Here the tail vectors of these chains are connected by the relations

$$f'_{m_1} = f_{n_2}; \quad f'_{m_2} = f_{n_3}; \dots, f'_{m_{k-1}} = f_{n_k}. \quad (7)$$

If $s = 0$, then these relations completely determine the unclosed object. If $s = 1$, then in the first chain the tail vector f_{n_1} is equal to zero, i.e. in this case $d_+ e_{n_1} = 0$. If $m_k = -1$, then in the k -th chain the vector f_{0k} is equal to zero, i.e. in this case the vector e_{1k} has no preimage with respect to the operator.

Assertion 3. *Let B and B' be two unclosed indecomposable objects, and let them be specified by the sets of numbers $(s, n_1, m_1; n_2, m_2; n_3, m_3; \dots; n_{k-1}, m_{k-1}; n_k, m_k)$ and $(s', n'_1, m'_1; n'_2, m'_2; \dots; n'_{k-1}, m'_{k-1}; n'_k, m'_k)$. The objects B and B' are equivalent (coincide) if and only if $s = s'$, $n'_i = n_i$, $m'_i = m_i$.*

A closed object in the simple case can be obtained from a single unclosed object determined by a set of numbers with $s = 0$, $n_1 > 0$; $m_k > 0$. In this case the vectors f_{n_1} and f'_{m_k} are nonzero. We shall call them the initial and terminal vectors of the unclosed object. The closed simple object is determined if, to the relations (7), one more is added,

$$f'_{m_k} = \mu f_{n_1}, \quad (8)$$

where μ is an arbitrary complex number.

Thus, in the simple case a closed object is specified by a set of numbers

$$(n_1, m_1; n_2, m_2; \dots; n_k, m_k; \mu),$$

where $n_i, m_i > 0$.

In the general case a closed object is specified by a set of numbers

$$(n_1, m_1; n_2, m_2; \dots; n_k, m_k; \mu; N),$$

where $n_i, m_i, N > 0$. This object is composed of unclosed objects, each of which is determined by the same set of numbers

$$(0, n_1, m_1; n_2, m_2; \dots; n_k, m_k).$$

Denote the initial and terminal vectors of the j -th ($j = 1, \dots, N$) unclosed object by f_j and f'_j . Then the closed object is determined by specifying the following relations between these vectors:

$$f'_1 = \mu f_1; \quad f'_2 = \mu f_2 + f_1; \dots; f'_i = \mu f_i + f_{i-1}, \dots \quad (i = 2, 3, \dots, N). \quad (9)$$

Assertion 4. *Let B and B' be two unclosed indecomposable objects from the category S_0 , and let them be specified by the sets of numbers $(n_1, m_1; n_2, m_2; \dots; n_k, m_k, \mu, N)$ and $(n'_1, m'_1; \dots; n'_k, m'_k, \mu', N')$. The objects B and B' coincide if and only if $\mu = \mu'$, $N = N'$, and the sequence $\{n'_i, m'_i\}$ is a cyclic permutation of the sequence $\{n_i, m_i\}$.*

Theorem 4. *Every indecomposable object from the category S_0 is either unclosed or closed, and the constructions indicated above exhaust all indecomposable objects.*

Corollary. Let M be an indecomposable Harish-Chandra module over the Lie algebra L , let M belong to the special category $C(\lambda_1, \lambda_2)$, and let the space M be a vector space over the field of complex numbers.

Then there are two alternatives: either the object is described by a set of integers, or the object is described by a set of integers and by the complex number μ . In order that the modules M and M' be equivalent, it is necessary and sufficient that $\lambda_1 = \lambda'_1$, $\lambda_2 = \lambda'_2$, $\mu = \mu'$, and that the corresponding sets of numbers coincide.

It is quite remarkable that an indecomposable module from a special category can have a complex number as an invariant.

In the work of D. P. Zhelobenko², a representation of the Lorentz group was considered, called a representation of finite rank. There the representation which, in our terminology, is called special is completely analyzed. However, in the special case there, the problem is only reduced to a certain algebraic one. The system of invariants of an indecomposable special object and, in particular, the presence of the continuous parameter μ and the canonical form are absent there.

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CITED LITERATURE

¹ I. M. Gel'fand, R. A. Minlos, Z. Ya. Shapiro, *Representations of the Rotation Group and of the Lorentz Group*, Moscow, 1958.

² D. P. Zhelobenko, DAN, **126**, No. 5, 935 (1959).

Note: Figure translations are in progress. See original paper for figures.

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