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Abstract

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MATHEMATICS

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PARTIAL SUFFICIENCY AND UNBIASED ESTIMATION OF POLYNOMIALS OF A SHIFT PARAMETER

(Presented by Academician Yu. V. Linnik on 26 VII 1966)

1. Let $\{P_\theta; \theta \in \Theta\}$ be a family of distributions on the space $(X, \mathfrak{A})$, depending on an abstract parameter $\theta \in \Theta$. A statistic $T(x)$ is called **sufficient for the family** $\{P_\theta\}$ if, for every bounded function $\varphi(x)$,

$$E_\theta(\varphi | T) = \tilde{\varphi} \text{ a.s. } P_\theta, \quad \theta \in \Theta, \quad (1)$$

for some $\tilde{\varphi} = \tilde{\varphi}(T(x))$. The answer to the question of when the statistic $T(x)$ is sufficient for the family $\{P_\theta\}$ is given by the factorization theorem ⁽¹⁾. The fundamental role of sufficient statistics in estimation theory is determined by the Rao–Blackwell–Kolmogorov theorem ^(2–4).

Of greatest interest is the case when the family $\{P_\theta\}$ of distributions in R^n is generated by a repeated sample $(x_1, \dots, x_n)$ from a one-dimensional population with distribution function (d.f.) $F(x; \theta)$,

$$P_\theta(A) = \int \dots \int_{(A)} dF(x_1; \theta) \dots dF(x_n; \theta). \quad (2)$$

For such families, necessary and sufficient conditions for the existence of non-trivial sufficient statistics, which are naturally formulated in the language of the one-dimensional d.f. $F(x; \theta)$, can be found in ⁽⁵⁾.

2. Yu. V. Linnik proposed the following definition of partial sufficiency of a statistic $T(x)$. Let $\mathcal{L} = \{\varphi\}$ be some linear system of functions on $(X, \mathfrak{A})$ for which $E_\theta|\varphi| < \infty$, $\theta \in \Theta$. A statistic $T(x)$ is called **$\mathcal{L}$ -sufficient** if condition (1) is fulfilled for every $\varphi \in \mathcal{L}$. The analogue of the Rao–Blackwell–Kolmogorov theorem has the following form.

If, for $\varphi \in \mathcal{L}$, $E_\theta\varphi^2 < \infty$, $\theta \in \Theta$, and $T(x)$ is an $\mathcal{L}$ -sufficient statistic, then for the function $\tilde{\varphi} = E_\theta(\varphi | T)$ we have

$$E_{\theta}\tilde{\varphi} = E_{\theta}\varphi, \quad E_{\theta}(\tilde{\varphi} - E_{\theta}\tilde{\varphi})^2 \leq E_{\theta}(\varphi - E_{\theta}\varphi)^2, \quad \theta \in \Theta.$$

In other words, all estimates from $\mathcal{L}$ are inadmissible, except, perhaps, those that depend only on the $\mathcal{L}$ -sufficient statistic.

Let us note that if P_{θ} has the form (2) and $\int x^{2k} dF(x; \theta) < \infty$, $\theta \in \Theta$, then the system $\mathcal{L}$ for which it is natural to seek partially sufficient statistics is the collection of all polynomials in $x_1, \dots, x_n$ of degree not exceeding k .

3. In what follows we shall assume that $(x_1, \dots, x_n)$ is a repeated sample from a population with d.f. $F(x - \theta)$, depending on a shift parameter $\theta \in R^1$, and

$$P_{\theta}(A) = \int_A \dots \int dF(x_1 - \theta) \dots dF(x_n - \theta).$$

It is well known ^(5,6) that if $F(x)$ is absolutely continuous with respect to Lebesgue measure, and the statistic

$$\bar{x} = \frac{1}{n}(x_1 + \dots + x_n), \quad n \geq 2,$$

is sufficient for the family $\{P_{\theta}\}$, then $F(x)$ is the distribution function of the normal law. As the result of note ⁽⁷⁾ shows, it is not possible to enlarge the family of normal distributions by replacing the sufficiency of the statistic $\bar{x}$ by its partial sufficiency in the sense of item 2. We shall modify the concept of partial sufficiency in the spirit of ⁽⁸⁾.

Suppose that

$$\int x^{2k} dF(x) < \infty \tag{3}$$

for some integer $k \geq 0$. When condition (3) is satisfied, the totality of all polynomials $\pi(x_1, \dots, x_n)$ of degree not higher than k forms a Hilbert space $L_k^{(2)}$, if the inner product of elements π_1 and π_2 is introduced in the usual way:

$$(\pi_1, \pi_2)_{\theta} = E_{\theta}(\pi_1 \pi_2).$$

By T_k we shall denote the subspace of L_k^2 generated by the functions

$$a_0 \bar{x}^k + \dots + a_k.$$

We shall say that T_k is an L_k^2 -sufficient subspace if, for any $\pi \in L_k^2$,

$$\hat{E}_{\theta}(\pi | T_k) = \tilde{\pi} \tag{4}$$

for some $\tilde{\pi} \in \tilde{T}_k$, where $\hat{E}_{\theta}(\cdot | T_k)$ is the projection operator onto T_k , when the inner product $(\cdot, \cdot)_{\theta}$ is introduced by means of the measure P_{θ} .

Theorem 1. *If the first $2k$ moments of the distribution function $F(x)$ coincide with the corresponding moments of some normal law, then T_k is an $L_k^{(2)}$ -sufficient subspace.*

Proof. Let first $F(x)$ be the distribution function of a normal law, and let $\pi(x_1, \dots, x_n) = \pi \in L_k^{(2)}$. Obviously, π can be represented in the form

$$\pi = \bar{x}^k q_0(x_2 - x_1, \dots, x_n - x_1) + \dots + q_k(x_2 - x_1, \dots, x_n - x_1),$$

where $q_j(y_2, \dots, y_n)$, $j = 0, \dots, k$, are also polynomials in $y_2, \dots, y_n$. But if $x_1, \dots, x_n$ are normally distributed random variables, then $\bar{x}$ and the vector $(x_2 - x_1, \dots, x_n - x_1)$ are independent. Therefore

$$E_\theta(\pi | \bar{x}) = a_0 \bar{x}^k + \dots + a_k, \quad (5)$$

where

$$a_j = E_\theta(q_j | \bar{x}) = E_\theta q_j = E_0 q_j. \quad (6)$$

Since $E_\theta(\pi | \bar{x}) \in T_k$, it follows that $\hat{E}_\theta(\pi | T_k) = E_\theta(\pi | \bar{x})$, and, according to (5), T_k is an $L_k^{(2)}$ -sufficient subspace if only $F(x)$ is the distribution function of a normal law. But two distributions whose first $2k$ moments are identical induce one and the same inner product in $L_k^{(2)}$. Therefore, for a distribution function $F(x)$ satisfying the condition of Theorem 1, we shall have

$$\hat{E}_\theta(\pi | T_k) = a_0 \bar{x}^k + \dots + a_k,$$

where $a_0, \dots, a_k$ are the same as in (6). Theorem 1 is proved.

4. The polynomial $\pi(x_1, \dots, x_n)$ is an unbiased estimate of the function

$$\bar{\pi}(\theta) = E_\theta \pi = c_0 \theta^k + \dots + c_k$$

with finite, if condition (3) is satisfied, variance for all $\theta \in R^1$. Let the quality measure of estimates be their variance. Then from Theorem 1 we obtain the following analogue of the Rao-Blackwell-Kolmogorov theorem.

Theorem 2. *If the first $2k$ moments of the distribution function $F(x)$ coincide with the corresponding moments of some normal law, then every polynomial $\pi(x_1, \dots, x_n) = \pi \in L_k^{(2)} \setminus T_k$ is inadmissible in the class of unbiased estimates of the function $\bar{\pi}(\theta)$.*

The question of whether estimators $\pi \in T_k$ are admissible or not requires special study. However, it is the optimal unbiased estimator of the function $\bar{\pi}(\theta)$ for all $\theta \in R^1$ in one single case, as the following result shows.

Theorem 3. *Let the distribution function $F(x)$ satisfy condition (3), and let the polynomial*

$$\pi(\bar{x}) = a_0 \bar{x}^k + \dots + a_k, \quad a_0 \neq 0,$$

of degree $k \geq 1$, for some $n \geq 3$, be the best unbiased estimator, for all $\theta \in R^1$, of the function

$$\pi(\theta) = c_0\theta^k + \dots + c_k.$$

Then $F(x)$ is the distribution function of the normal law.

As shown in (9), for $k = 1$ Theorem 3 remains valid if one assumes only the admissibility of the estimator $a_0\bar{x} + a_1$.

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REFERENCES

1. P. Halmos, L. Savage, Ann. Math. Stat., **20**, 2 (1949).
2. C. R. Rao, Bull. Calcutta Math. Soc., **37** (1945).
3. D. Blackwell, Ann. Math. Stat., **18**, 1 (1947).
4. A. N. Kolmogorov, Izv. AN SSSR, ser. matem., **14**, 4 (1950).
5. E. B. Dynkin, UMN, **6**, 1 (1951).
6. Ferguson, Ann. Math. Stat., **33**, 3 (1962).
7. A. M. Kagan, Yu. V. Linnik, *Characterization Problems in Probability Theory and Their Applications*, **12**, 2 (1967).
8. A. M. Kagan, Tr. Matem. inst. im. V. A. Steklova AN SSSR, **79**, 17 (1965).
9. A. M. Kagan, Yu. V. Linnik, C. R. Rao, Sankhyā (1965).

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