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Abstract

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MATHEMATICS

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ASYMPTOTICS OF THE NUMBER OF SOLUTIONS OF CERTAIN DIOPHANTINE INEQUALITIES

(Presented by Academician I. M. Vinogradov on 30 V 1966)

Let α, β be real numbers; let $\chi(q)$ be a positive monotonically decreasing function defined for natural numbers q ; let $\|x\| = \min(\{x\}, 1 - \{x\})$ be the distance from x to the nearest integer. It is known that for almost all α the number of solutions of the inequality

$$\|\alpha q + \beta\| < \chi(q) \quad (1)$$

in integers q , $1 \leq q \leq Q$, is asymptotically equal to $2\Psi(Q) + O(\Psi^{1/2+\varepsilon}(Q))$, where $\Psi(Q) = \sum_{q=1}^Q \chi(q)$, if $\Psi(Q) \rightarrow \infty$ (see, for example, ^(5, 8, 11, 12)). It is of interest to give "individual" variants of this fact, i.e. to indicate concrete numbers α (or classes of numbers α) for which the analogous assertion is true. Recently S. Lang ^(6, 7) proved for quadratic irrationalities α and functions $\chi(q)$ decreasing as q^{-1} or considerably more slowly than q^{-1} , that the number of solutions of (1) with $\beta = 0$ has the asymptotic $2\Psi(Q)$. Lang's arguments are based on the arithmetic of quadratic fields and do not admit any generalization to the case of other α .

In this note we show, in particular, that if α satisfies the following condition (A): the number of solutions of the inequality $\|\alpha q\| < q^{-1-\varepsilon}$ is finite for every $\varepsilon > 0$, then for any β and any σ , $0 < \sigma < 1$, the number of solutions of the inequality

$$\|\alpha q + \beta\| < q^{-\sigma}, \quad 1 \leq q \leq Q, \quad (2)$$

is asymptotically equal to

$$\frac{2}{1-\sigma} Q^{1-\sigma} + O(Q^{(1-\sigma)/2+\varepsilon}).$$

The following more general result is true.

Theorem. Let $\chi(t)$ be a positive continuously differentiable function defined for $t \geq 1$, whose derivative satisfies the conditions: 1) $\chi'(t) < 0$ for all $t \geq 1$, $|\chi'(t)|$ tends monotonically to zero as $t \rightarrow \infty$; 2) $|\chi'(t)| \gg t^{-2}$ ($t \gg 1$), and if $t < t_1 \leq t$, then $|\chi'(t)| \ll |\chi'(t_1)|$. Then, for α satisfying condition (A), and for any β , the number of solutions of inequality (1) is asymptotically equal to

$$2\Psi(Q) + O(\Psi(Q)\chi^{-1}(Q)|\chi'(Q)|^{1/2}Q^\varepsilon) + O(\chi(Q)|\chi'(Q)|^{-1/2}),$$

where

$$\Psi(Q) = \int_1^Q \chi(t) dt.$$

This theorem is an immediate consequence of the well-known fact of the strong uniform distribution of the fractional parts $\|\alpha q + \beta\|$ for α satisfying condition (A) (see Lemma 1), and of Lemma 2 below, which makes it possible to determine the asymptotics of the number of solutions of the inequality $a_q < \chi(q)$, if it is known that the sequence a_q is well distributed on some interval $(0, \lambda)$, $\lambda > 0$.

It is known that condition (A) is satisfied by irrational algebraic numbers $(^{10}, ^2)$, and also by numbers of the form e^r ($r \neq 0$) and $\ln r$ ($r \neq 0, 1$), where r is a rational number $(^3, ^4, ^9)$.

Lemma 1. Let α be a real number satisfying condition (A); let β be any real number; let I be an arbitrary interval inside the interval $(0, \frac{1}{2})$; and let $N_Q(I)$ be the number of fractional parts $\|\alpha q + \beta\|$, $1 \leq q \leq Q$, lying in I . Then

$$N_Q(I) = 2|I|Q + O(Q^\varepsilon),$$

where $|I|$ is the length of the interval I , and the estimate is uniform over all intervals I .

The proof of this lemma is obtained without difficulty by applying I. M. Vinogradov's "little cups" method $(^{(1)}, \text{p. 260})$.

Lemma 2. Let a_q ($q = 1, 2, \dots$) be a sequence of real numbers lying in the interval $(0, \lambda)$, where $\lambda > 0$ is some fixed number; let I be an arbitrary interval inside the interval $(0, \lambda)$; let $N_Q(I)$ be the number of those a_q , $1 \leq q \leq Q$, which lie in I , and suppose that

$$N_Q(I) = \lambda^{-1}|I|Q + R_Q(I), \quad |R_Q(I)| \leq R(Q), \quad (3)$$

where the quantity $R(Q)$ does not depend on I , and $R(Q)$ increases monotonically to infinity. Finally, let $\chi(t)$ be a function satisfying the conditions of the theorem. Then the number $N(Q)$ of those a_q which satisfy the inequality $a_q < \chi(q)$, $1 \leq q \leq Q$, is equal to

$$\lambda^{-1}\Psi(Q) + L(Q),$$

$$L(Q) = O(\Psi(Q)\chi^{-1}(Q)|\chi'(Q)|^{1/2}R(Q) + \chi(Q)|\chi'(Q)|^{-1/2}).$$

Proof. Define the numbers p_k from the conditions $p_1 = 1$,

$$p_{k+1} - p_k = |\chi'(p_k)|^{-1/2}.$$

It is clear that the numbers p_k are positive and the difference $p_{k+1} - p_k$ increases monotonically. Denote by $N_0(k)$ the number of those a_q for which the inequality $a_q < \chi(q)$ holds under the condition $p_k \leq q < p_{k+1}$; by $N_1(k)$, the number of those a_q for which $a_q < \chi(p_k)$ under $p_k \leq q \leq p_{k+1}$; and, analogously, denote by $N_2(k)$ the number of those a_q for which $a_q < \chi(p_{k+1})$ under $p_k \leq q < p_{k+1}$. Since the function $\chi(t)$ decreases monotonically, we have $N_2(k) \leq N_0(k) \leq N_1(k)$, i.e.

$$N_0(k) = N_1(k) - \theta(N_1(k) - N_2(k)), \quad 0 \leq \theta \leq 1.$$

Since the difference $N_1(k) - N_2(k)$ is the number of those a_q which fall into the interval $(\chi(p_{k+1}), \chi(p_k))$, by the condition of the lemma (3) this difference does not exceed the quantity

$$O((\chi(p_k) - \chi(p_{k+1}))(p_{k+1} - p_k) + R(p_{k+1})).$$

Define k_0 from the condition $p_{k_0} \leq Q < p_{k_0+1}$. Then, since

$$N_1(k) = \lambda^{-1}\chi(p_k)(p_{k+1} - p_k) + O(R(p_{k+1})),$$

we have

$$\begin{aligned} N(Q) &= \sum_{k=1}^{k_0-1} N_1(k) + L_1(Q) = \\ &= \sum_{k=1}^{k_0-1} \lambda^{-1}\chi(p_k)(p_{k+1} - p_k) + L_2(Q) = \lambda^{-1}\Psi(Q) + L_3(Q), \end{aligned}$$

where all quantities $L_j(Q)$ have order

$$O\left(\sum_{k=1}^{k_0-1} (\chi(p_k) - \chi(p_{k+1}))(p_{k+1} - p_k) + k_0 R(Q) + \chi(p_{k_0})(Q - p_{k_0})\right). \quad (4)$$

Let x, y be real numbers satisfying $p_k \leq x < y \leq p_{k+1}$. By the choice of the numbers p_k and the monotone variation of the functions $\chi(t)$, $|\chi'(t)|$, we find

$$\begin{aligned} 0 &< (\chi(x) - \chi(y))(y - x) \leq |\chi'(x)|(y - x)^2 \leq \\ &\leq |\chi'(p_k)|(p_{k+1} - p_k)(y - x) = (y - x)(p_{k+1} - p_k)^{-1} \leq 1. \end{aligned}$$

Therefore

$$\chi(p_{k_0})(Q - p_{k_0}) \leq \chi(Q)(Q - p_{k_0}) + 1 \leq \chi(Q)|\chi'(Q)|^{-1/2} + 1$$

and the remainder term (4) will be a quantity

$$O(k_0 R(Q) + \chi(Q) |\chi'(Q)|^{-1/2}).$$

Let us estimate the quantity k_0 . Setting $k_1 = [k_0/2]$, we have

$$\begin{aligned} \int_{q_{k_1}}^{q_{k_0}} \chi(t) dt &\geq \chi(Q)(p_{k_0} - p_{k_1}) = \chi(Q) \sum_{k_1 \leq k < k_0} (p_{k+1} - p_k) = \\ &= \chi(Q) \sum_{k_1 \leq k < k_0} |\chi'(p_k)|^{-1/2} \geq \chi(Q) |\chi'(p_{k_1})|^{-1/2} (k_0 - k_1). \end{aligned}$$

By virtue of the conditions imposed on $|\chi'(t)|$, we find

$$\frac{p_{k+1}}{p_k} = 1 + p_k^{-1} |\chi'(p_k)|^{-1/2} \ll 1, \quad p_{k+1} \ll p_k;$$

therefore $Q \ll p_{k_1}$, and then $|\chi'(p_{k_1})| \ll |\chi'(Q)|$. Thus,

$$k_0 \ll \chi^{-1}(Q) |\chi'(Q)|^{1/2} \int_1^Q \chi(t) dt,$$

which completes the proof of Lemma 2.

In the case of simultaneous approximations

$$\max(\|a_1 q + \beta_1\|, \dots, \|a_{nq} + \beta_n\|) < \chi(q) \quad (5)$$

analogous results hold, as is not difficult to establish by only slightly developing the arguments. Condition (A) here must be replaced by the condition: the number of solutions of the inequality

$$\|a_1 q\| \cdot \|a_2 q\| \dots \|a_{nq}\| < q^{-1-\varepsilon}$$

in integers $q > 0$, for every $\varepsilon > 0$, is finite; or, equivalently, the number of solutions of the inequality

$$\|a_1 a_1 + \alpha_2 a_2 + \dots + a_n a_n\| > (a'_1 a'_2 \dots a'_n)^{-n-\varepsilon}$$

in integers a_i ($i = 1, 2, \dots, n$), where $a'_i = \max(1, |a_i|)$, is finite for every $\varepsilon > 0$. For example, one may take $\alpha_i = e^{r_i}$, where $r_i \neq 0$ are pairwise distinct rational numbers ⁽⁴⁾. Under these conditions, the number of solutions of the system of

inequalities (5) with $\chi(q) = q^{-\sigma}$, $0 < \sigma < 1/n$, $1 \leq q \leq Q$, is asymptotically equal to

$$\frac{2^n}{1 - n\sigma} Q^{1-n\sigma} + O\left(Q^{\frac{1-n\sigma}{2} + \varepsilon}\right).$$

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