

CONSTRUCTIVE GENERALIZED FUNCTIONS

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Abstract

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MATHEMATICS

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CONSTRUCTIVE GENERALIZED FUNCTIONS

(Presented by Academician P. S. Novikov on 22 X 1966)

1. The present note is devoted to defining the concept of a generalized function in constructive mathematics. First of all we introduce some concepts and notation*.

By a **word of type** ^{k} we shall mean the complete code of a uniformly continuous function on the interval $-k\Delta k$, where k is an arbitrary natural number greater than 0. By C_k we shall denote the normed space of words of type ^{k} . Let us note that if $k \leq l$, then every word of type ^{l} is a word of type ^{k} . The equality relation, the addition algorithm, and the algorithm of multiplication by real duplexes in the space C_k will be denoted respectively by

$$=, +, \cdot$$

Let f be an everywhere defined constructive function. We shall say that f is an **almost uniformly continuous function** if there is an algorithm g of type $(\rightarrow)$ such that, for every $k \geq 1$, g_k is a regulator of uniform continuity of the function f on the interval $-k\Delta k$.

The algorithm g is called a **regulator of almost uniform continuity** of the function f . The word $\xi f \bar{1} \xi g$ will be called the **complete code of the almost uniformly continuous function**, and also a word of type $\cdot$. One can define an equality relation $=$, an addition algorithm $+$, and an algorithm of multiplication by real duplexes $\cdot$ in the usual way, so that the set of words of type $\cdot$ forms a constructive linear space. We denote this space by $\mathcal{C}$. It is not difficult to construct an algorithm ξ having the following property: for every $k \geq 1$, $\tilde{\xi}_k$ transforms any word of the form $\xi f \bar{1} \xi g$ into the word $\xi f \bar{1} \xi \tilde{g}_k$. Obviously, ξ_k transforms every word of type $\cdot$ into a word of type ^{k} . Let 1 and 2 be words of type $\cdot$. Instead of the formula

$$\xi(k \square_1) = \xi(k \square_2)$$

we shall simply write

$$1 = 2$$

We can construct an algorithm J_1 , which, when applied to any word of the form $k \square x \square y \square \frac{k}{1}$, where k is a natural number ≥ 1 , x and y are real duplexes, $-k \leq x \leq y \leq k$, and $\frac{k}{1}$ is a word of type $\frac{k}{1}$, transforms every such word into the integral of the complete code $\frac{k}{1}$ in the space $\mathfrak{L}_1(x, y)$ (see ⁽²⁾, § 15.5). Then we construct an algorithm J such that, for every $k \geq 1$, J_k is an algorithm of type $(\frac{k}{1} \rightarrow \frac{k}{1})$ and such that, for any number $k \geq 1$, any word $\frac{k}{1}$ of type $\frac{k}{1}$, and any duplex x , $-k \leq x \leq k$, the following holds:

$$\underbrace{J(k \square \frac{k}{1})}_{,k}(x) \simeq J_1(k \square \min(x \square 0) \square x \square \frac{k}{1}) - J_1(k \square \min(x \square 0) \square 0 \square \frac{k}{1}).$$

* All specifically unexplained terms and notation are understood in the same way as in ⁽¹⁻³⁾.

The number $J(k \square \mathfrak{sh}_1^k)(x)$ is called the **integral** of the complete cipher $\mathfrak{sh}_1^k$ from 0 to x , and $J(k \square \mathfrak{sh}_1^k)$ is called the **primitive** of the complete cipher $\mathfrak{sh}_1^k$ on the segment $-k \Delta k$. Now we can construct an algorithm I such that, for every $k \geq 1$, I_k is an algorithm of type $(\mathfrak{sh}_1^k \rightarrow \mathfrak{sh}_1^k)$ and the following scheme is satisfied:

$$I(k \square 0 \square \mathfrak{sh}_1^k) \simeq \mathfrak{sh}_1^k; \quad I(k \square n + 1 \square \mathfrak{sh}_1^k) \simeq J(k \square I(k \square n \square \mathfrak{sh}_1^k)).$$

$I(k \square n \square \mathfrak{sh}_1^k)$ is called the **primitive of order n** of the complete cipher $\mathfrak{sh}_1^k$ on the segment $-k \Delta k$.

After this one can construct an algorithm $\mathfrak{I}$ of type $(\mathfrak{sh}_1 \rightarrow \mathfrak{sh}_1)$, satisfying the following condition for any numbers k, n ($k \geq 1$) and any word $\mathfrak{sh}_1$ of type $\mathfrak{sh}_1$:

$$\xi(k \square \mathfrak{I}(n \square \mathfrak{sh}_1)) = I(k \square n \square \xi(k \square \mathfrak{sh}_1)).$$

$\mathfrak{I}(n \square \mathfrak{sh}_1)$ is called the **primitive of order n** of the complete cipher $\mathfrak{sh}_1$.

We shall call every σ -system of real duplexes a **polynomial germ**, and also a word of type $\mathfrak{yu}$ ⁽²⁾. Let L denote an algorithm computing the number of σ -terms in a word of type $\mathfrak{yu}$, and let G denote an algorithm of type $(\mathfrak{yu} \rightarrow \mathfrak{sh})$ having the following property: whatever the word $\mathfrak{yu}_1$,

$$\forall_x (G(\mathfrak{yu}_1)(x) \simeq a_1 + a_2 \cdot x + \dots + a_n \cdot x^{n-1}),$$

where $n \simeq L(\mathfrak{yu}_1)$, and a_i is the i -th σ -term of the word $\mathfrak{yu}_1$ ($i = 1, \dots, n$). We note that all judgments constructed with the aid of the indicated concepts and notation can be formulated rigorously within the framework of the logico-mathematical languages of N. A. Shanin ⁽¹⁾.

2. We now define the concept of a constructive generalized function. In order to define this concept, we shall rely on the classical definition of Mikusinski–Sikorski–Korevaar ^(7,8).

Of an algorithm λ of type $(\mathbf{n} \rightarrow \mathbf{sh})$ we shall say that it is a **fundamental sequence of words of type sh** if potentially realizable algorithms N of type $(\mathbf{n} \rightarrow \mathbf{n})$, P of type $(\mathbf{nn} \rightarrow \mathbf{nu})$, and Q of type $(\mathbf{nn} \rightarrow \mathbf{n})$ exist, satisfying the following conditions:

$$1) \quad \forall kn(L(P(k \square n)) < N(k))^*;$$

$$2) \quad \forall k i j m (|x| \leq k \ \& \ i, j \geq Q(k \square m) \supset |F(k \square i)(x) - F(k \square j)(x)| < \bar{2}^m).$$

where F is an algorithm of type $(\mathbf{nn} \rightarrow \mathbf{sh})$, defined as follows:

$$\forall kn(F(k \square n) \simeq J(N(k) \square \lambda(n)) + G(P(k \square n))).$$

Of a word X in the alphabet $\mathfrak{Ch}_0$ we shall say that it is an **F -generalized function** if the algorithm $\langle X \rangle$ is a fundamental sequence of words of type $\mathbf{sh}$.

Of a word A in the alphabet $\mathfrak{Ch}_0 \cup \{t\}$ we shall say that it is an **FR -generalized function** if it has the form $XtYtZtT$, where X is a record of some algorithm λ of type $(\mathbf{n} \rightarrow \mathbf{sh})$, Y is a record of some algorithm N of type $(\mathbf{n} \rightarrow \mathbf{n})$, Z is a record of some algorithm P of type $(\mathbf{nn} \rightarrow \mathbf{nu})$, T is a record of some algorithm Q of type $(\mathbf{nn} \rightarrow \mathbf{n})$, and conditions 1)–2) are satisfied.

An FR -generalized function we shall also call a **constructive generalized function** or a word of type Φ . It is easy to see that the set of constructive generalized functions can be defined by a normal formula.

* The notation $<$ is understood in the following way:

$$(L(\mathbf{nu}_1) < n) \Leftrightarrow (n \geq 1 \supset L(\mathbf{nu}_1 \leq n) \ \& \ (n = 0 \supset \mathbf{nu}_1 = 0)),$$

where $\mathbf{nu}_1$ is a word of type $\mathbf{nu}$ and n is a natural number.

Let $\Phi_1 \varpi \xi \lambda \mathcal{E} N \mathcal{E} P \mathcal{E} Q \mathcal{E}$ be a constructive generalized function. We agree to denote:

$$|\Phi_1|_k \varpi N(k); \quad [\Phi_1]_{k,n} \varpi P(k \square n).$$

According to condition 2), the algorithm $(\xi_k \circ \bar{F}_{k \square})$ determines a sequence of words of type ^{k} converging in itself, with convergence regulator $Q_{k \square}$. One can construct an algorithm that transforms each word of the form $k \square \Phi_1$ into the limit of the sequence $(\xi_{k \square} \circ \bar{F}_{k \square})$ in the space C_k . We denote this limit by $\{\Phi_1\}_k$. This is a word of type ^{k} .

Let two constructive generalized functions Φ_1 and Φ_2 be given. Denote

$$p_k \varpi |\Phi_1|_k, \quad q_k \varpi |\Phi_2|_k, \quad l_k \varpi \max(p_k \square q_k).$$

We shall say that Φ_1 is equal to Φ_2 and shall write $\Phi_1 = \Phi_2$, if there is an algorithm $\mathcal{P}$ of type $(\rightarrow)$ satisfying the conditions:

- 1) $\forall k(\bar{L}(\mathcal{P}(k)) < l_k)$;
- 2) $\forall k(I(k \square l_k - p_k \square \{\Phi_1\}) = I(k \square l_k - q_k \square \{\Phi_2\}_k) + \xi(k \square G(\mathcal{P}(k))))$.

Theorem 1. *There is an algorithm that transforms each pair of equal constructive generalized functions Φ_1 and Φ_2 into a notation of some algorithm $\mathcal{P}$ of type $(\rightarrow)$ satisfying conditions 1)–2).*

It follows from Theorem 1 that the relation of equality between constructive generalized functions can be defined by a normal formula.

Let $\Phi_1 \rightleftharpoons X\tau Y\tau Z\tau T$ be a constructive generalized function. The word X will be called the **basis** of the generalized function Φ_1 . We shall say that the constructive generalized function Φ_1 is equal to the F -generalized function X , if it is equal to some constructive generalized function whose basis is X .

Theorem 2. *a) For every F -generalized function one can construct an equal constructive generalized function; b) there is no algorithm that transforms every F -generalized function into an equal constructive generalized function.*

It is not difficult to define addition of constructive generalized functions, multiplication of constructive generalized functions by real duplexes, and the zero constructive generalized function in such a way that the set of constructive generalized functions forms a linear space (see the definition in (6)).

3. We now define the derivative of a constructive generalized function.

Let $\Phi_1 \rightleftharpoons X\tau Y\tau Z\tau T$ be a constructive generalized function. Construct algorithms γ and δ of type $(\rightarrow)$, such that for any natural numbers k, n :

$$\gamma(n) \simeq \langle X \rangle(n)(n \square n); \quad \delta(k) \simeq \mu j \left(2^j > \frac{4k^{|\Phi_1|_k}}{|\Phi_1|_{k-1}!} \right).$$

After this, construct algorithms λ of type $(\rightarrow)$, N of type $(\rightarrow)$, P of type $(\rightarrow)$, and Q of type $(\rightarrow)$, satisfying the following conditions for any natural numbers k, n, m and any duplex x :

$$\lambda(n)(x) \simeq 2^{\gamma(n)}(\langle X \rangle(n)(x + 2^{-\gamma(n)}) - \langle X \rangle(n)(x)); \quad N(k) \simeq |\Phi_1|_k + 1;$$

$$P(k \square n) \simeq [\Phi_1]_{k,n} + \underbrace{0\sigma \dots \sigma 0\sigma}_{|\Phi_1|_k \text{ times}} \frac{\langle X \rangle(n)(0)}{|\Phi_1|_k!};$$

$$Q(k \square m) \simeq \max(m + 1 + \delta(k) \square k + 1 \square \langle T \rangle(k \square m + 1)).$$

Put

$$X_1 \rightleftharpoons \mathcal{E}\lambda\mathcal{E}, \quad Y_1 \rightleftharpoons \mathcal{E}N\mathcal{E}, \quad Z_1 \rightleftharpoons \mathcal{E}P\mathcal{E}, \quad T_1 \rightleftharpoons \mathcal{E}Q\mathcal{E}.$$

It is easy to prove that the word $X_1\tau Y_1\tau Z_1\tau T_1$ is a constructive generalized function. We shall call this generalized function the **derivative** of the constructive generalized function Φ_1 . One can construct an algorithm D ,

constructing, for each constructive generalized function, its derivative. After this one can construct an algorithm $\mathfrak{D}$ of type $(\mathfrak{nf} \rightarrow \Phi)$ satisfying the scheme:

$$\mathfrak{D}(n\Box\Phi_1) \simeq \Phi_1; \quad \mathfrak{D}(n+1\Box\Phi_1) \simeq D(\mathfrak{D}(n\Box\Phi_1)).$$

$\mathfrak{D}(n\Box\Phi_1)$ is called the derivative of n -th order of Φ_1 . Thus every constructive generalized function has derivatives of arbitrary order.

4. Denote by R an algorithm of type $(\mathfrak{m} \rightarrow \Phi)$ which transforms every word $\mathfrak{m}_1$ of type $\mathfrak{m}$ into a word $XtYtZtT$ of type Φ such that

$$\forall_n(\langle X \rangle(n) \doteq \mathfrak{m}_1); \quad \forall_k(\langle Y \rangle(k) \doteq 0);$$

$$\forall_{kn}(\langle Z \rangle(k\Box n) \doteq 0); \quad \forall_{km}(\langle T \rangle(k\Box m) \doteq 0).$$

We shall say that $R(\mathfrak{m}_1)$ is a representative of the complete cipher of an almost uniformly continuous function $\mathfrak{m}_1$ in the space of constructive generalized functions.

Let Φ_1 be a constructive generalized function. We shall say that Φ_1 is a generalized function of order $\leq n$, if

$$\exists \Phi_2(\Phi_1 = \Phi_2 \& \forall k(|\Phi_2|_k \geq n));$$

that Φ_1 is a generalized function of finite order if there exists a number n such that Φ_1 is a generalized function of order $\leq n$; that Φ_1 is a generalized function of n -th order if it is a generalized function of order $\leq n$ and, when $n \geq 1$, it is not a generalized function of order $\leq n-1$; that Φ_1 is a regular generalized function if it is a generalized function of order 0.

Theorem 3. It is not true that for every constructive generalized function Φ_1 of finite order there exists a number n such that Φ_1 is a constructive generalized function of n -th order.

Theorem 4. There is no algorithm which constructs, for every constructive generalized function Φ_1 of finite order, a number n such that Φ_1 is a constructive generalized function of order $\leq n$.

Theorem 5. There exists an algorithm which transforms every pair of the form $n\Box\Phi_1$, satisfying the condition $\forall k(|\Phi_1|_k \leq n)$, into a word $\mathfrak{m}_1$ of type $\mathfrak{m}$ such that

$$\Phi_1 = \mathfrak{D}(n\Box R(\mathfrak{m}_1)).$$

Roughly speaking, this means that every constructive generalized function of order $\leq n$ is the derivative of n -th order of some locally uniformly continuous function.

Corollary. For every regular constructive generalized function one can construct a word $\mathfrak{m}_1$ of type $\mathfrak{m}$ such that

$$\Phi_1 = R(\mathfrak{m}_1).$$

Theorem 6. Whatever the natural number n , there is no algorithm which transforms every constructive generalized function Φ_1 of order $\leq n$ into a constructive generalized function Φ_2 such that

$$\Phi_1 = \Phi_2 \& \forall k (|\Phi_2|_k \leq n).$$

Corollary. There is no algorithm which transforms every regular constructive generalized function Φ_1 into a word $\mathfrak{m}_1$ of type $\mathfrak{m}$ such that

$$\Phi_1 = R(\mathfrak{m}_1).$$

Theorem 3 is proved with the aid of Lemma 2, § 2 from (4). Theorem 6 is proved with the aid of Theorems 10.2.2 and 10.3.1 from (5).

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Note: Figure translations are in progress. See original paper for figures.

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