

ON OPEN MAPPINGS

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Abstract

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MATHEMATICS

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(Presented by Academician P. S. Aleksandrov on February 1, 1966)

I. In ⁽¹⁾ P. S. Aleksandrov showed that countably multiple open mappings of compacta do not increase their dimension, and in ⁽²⁾ A. N. Kolmogorov showed that the set of points of local topologicity of a countably multiple open mapping of a compactum X is everywhere dense in X . P. S. Aleksandrov raised the question of extending his theorem to arbitrary bcompacta. Below we give generalizations of the theorems of Aleksandrov and Kolmogorov; in particular, an answer is given to P. S. Aleksandrov's question.

All spaces under consideration are assumed to be completely regular, and all mappings are continuous mappings "onto." By a complete space is meant a space complete in the sense of Čech, i.e., a space that is a set of type G_δ in some (any) of its bcompact extensions. A space is locally complete if each of its points has a neighborhood whose closure (and hence the neighborhood itself) is complete.

By a σ -discrete set we mean a set that decomposes into the sum of a countable number of discrete (i.e., also closed) subsets. Obviously, every σ -discrete bcompactum (and even every finally compact space) is at most countable. A mapping $f : X \rightarrow Y$ will be called σ -discrete if every set $f^{-1}(y)$, $y \in Y$, is σ -discrete. The mapping f is locally σ -discrete if for each point $x \in X$ there exists a neighborhood on which the mapping f is σ -discrete. Examples of σ -discrete mappings are countably multiple mappings.

We now formulate the first general result, which generalizes A. N. Kolmogorov's theorem.

Theorem 1. If the mapping $f : X \rightarrow Y$ is open and locally σ -discrete, and the space X is locally complete, then the set of points of local topologicity of the mapping f is everywhere dense (and open) in X .

Corollary 1. The set of points of local topologicity of an open countably multiple mapping of a complete (for example, locally bcompact) space X is everywhere dense and open in X .*

Corollary 2. The set of points of local topologicity of an open countably multiple mapping of a bcompactum X is everywhere dense in X .

We now formulate the second general result, which generalizes P. S. Aleksandrov's theorem.

Theorem 2. If a mapping f of a locally complete normal space X onto a weakly paracompact normal space Y is open and σ -discrete, then

$$\dim Y \leq \text{loc dim } X \leq \dim X^{**}.$$

Corollary 3. If a mapping f of a complete normal space X onto a weakly paracompact normal space Y is open

* After obtaining this result, V. Proizvolov generalized it somewhat, though only for the case of locally bicom pact spaces ⁽⁸⁾.

** $\dim X$ is defined by means of open finite coverings. For $\text{loc dim } X$ and $\text{loc Ind } X$, see ⁽³⁾.

and countably many, then

$$\dim Y \leq \dim X.$$

Corollary 4. If the mapping $f : X \rightarrow Y$ of a bicom pactum is open and countably many, then

$$\dim Y = \dim X.$$

Corollary 5. If the mapping $f : X \rightarrow Y$ of a complete paracompactum X is open-closed and σ -discrete, then

$$\dim Y = \dim X.$$

It follows from Corollary 1 that A. D. Taimanov's theorem ⁽⁴⁾ on the non-increase of dimension under countably many open mappings of complete spaces with a countable base follows, as does the later theorem of A. V. Arhangel'skii on the non-increase of the dimension of complete metric spaces under the same mappings onto metric spaces ⁽⁵⁾.

The following theorem also generalizes a theorem of P. S. Aleksandrov.

Theorem 3. If the mapping f of a locally complete normal space X onto a totally normal ⁽³⁾ weakly paracompact (for example, hereditarily paracompact, or weakly paracompact perfectly normal, or metric) space Y is open and locally σ -discrete, then

$$\text{Ind } X \geq \text{Ind } Y.$$

Corollary 6. If the mapping f of a locally complete hereditarily paracompact space X onto a space Y is open-closed and locally σ -discrete, then

$$\text{Ind } X = \text{Ind } Y.$$

Corollary 7. If the mapping f of a bicompactum X onto a bicompactum Y is open and countably many, and at least one of the bicompacta X or Y is perfectly normal, then

$$\text{Ind } X = \text{Ind } Y.$$

Theorem 3 can be somewhat generalized by replacing in it the totally normal space Y by a Dowker space: a hereditarily normal space will be called **Dowker** if every open subset of X has a point-finite open covering by sets of type F_σ in X . Examples of Dowker spaces are hereditarily normal hereditarily weakly paracompact spaces. Dowker spaces behave, with respect to the dimensions Ind and dim , in the same way as totally normal spaces.

Theorem 4. If the space X is Dowker and $A \subseteq X$, then

$$\text{dim } A \leq \text{dim } X, \quad \text{Ind } A \leq \text{Ind } X.$$

Theorem 5. If for a Dowker space X one of the conditions is fulfilled: a) $X = A \cup B$, $\text{Ind } A \leq n$, $\text{Ind } B \leq n$, and the set A is closed in X ; b) $X = \bigcup_{i=1}^{\infty} X_i$, $\text{Ind } X_i \leq n$, and the sets X_i are closed in X , then

$$\text{Ind } X \leq n.$$

Theorem 6. For a weakly paracompact Dowker space the relation

$$\text{Ind } X = \text{loc Ind } X$$

holds.

Let us also note that the following is true.

Proposition 1. Let a normal subspace A of a normal space X be such that there exists a point-finite covering of the set A by sets open in A and of type F_σ in X ; then

$$\text{dim } A \leq \text{dim } X.$$

II. In ⁽⁶⁾ Hausdorff proved that, under an open mapping f of a metric space X onto a metric space Y , the completeness of X implies the completeness of Y . In ⁽⁷⁾ E. Michael showed that Hausdorff's theorem remains valid if Y is only paracompact. Moreover, he showed that in X there exists a set which is mapped perfectly by means of f onto Y (thus, Y nevertheless turns out to be metrizable). The following theorems generalize the results of Hausdorff and Michael.

Theorem 7. *If a mapping $f : X \rightarrow Y$ of a locally complete space X is open, then in X there is a subset X' which is mapped perfectly by means of f onto an everywhere dense subset Y' of Y . The set X' is closed in $f^{-1}(Y')$ and, in the case of a complete space X , is of type G_δ in X .*

Corollary 8. *Open images of locally complete spaces are extensions of complete spaces (they contain complete everywhere dense subspaces).*

Corollary 9. *An open image of a complete metric space is an extension of a complete metric space (and possesses the first axiom of countability).*

Theorem 8. *If a mapping $f : X \rightarrow Y$ of a locally complete space X is open, then for every paracompact $Y' \subseteq Y$ there exists in X a subset X' which is mapped perfectly by means of f onto Y' . The set X' is closed in $f^{-1}(Y')$ and, in the case of a complete space X , is of type G_δ in $f^{-1}(Y')$.*

Corollary 10. a) *If a paracompact Y is an open image of a locally complete space, then Y is a complete space;* b) *more generally, if a space Y is an open image of a locally complete space, then every paracompact set $Y' \subseteq Y$, closed or of type G_δ in Y , is complete.*

Corollary 11. *If a space Y is an open image of a complete metric space, then every paracompact subspace Y is metrizable, and if this subspace is closed or of type G_δ in Y , then it is completely metrizable.*

This corollary, however, also follows from the results of E. Michael.

We prove Theorem 8 for the case $Y' = Y$ and part a) of Corollary 10. We shall first suppose that the space X is complete. Denote by $\bar{f}$ the extension of the mapping f to a mapping of βX onto βY (βA denotes the maximal bicomact extension of A), and denote the set $\bar{f}^{-1}(Y)$ by $\bar{X}$. The mapping $\bar{f} : \bar{X} \rightarrow Y$ is perfect, and $\bar{X}$ is the intersection of a countable system of open sets O_i , $i = 1, 2, \dots$, in $\bar{X}$.

Choose for each point $y \in Y$ a point $x(y) \in f^{-1}(y)$ and its neighborhood $Vx(y)$ in X such that

$$Vx(y) \subseteq [Vx(y)]_{\bar{X}} \subseteq O_1.$$

The system v of the sets $V_y = f(Vx(y))$ is an open cover of Y (by virtue of the openness of f). Inscribe in the cover v a locally finite open cover

$$\omega = \{U_\alpha\}, \quad \alpha \in \mathfrak{A},$$

of the space Y . For each α we fix exactly one set $Vx(y) = V_\alpha$ for which $f(Vx(y)) \supseteq U_\alpha$. The system of open sets in X

$$W_\alpha = V_\alpha \cap f^{-1}(U_\alpha)$$

is locally finite in $\bar{X}$, for the locally finite cover

$$\omega^{-1} = \{\bar{f}^{-1}(U_\alpha)\}, \quad \alpha \in \mathfrak{A}$$

is locally finite in $\bar{X}$ (as the inverse image of a locally finite cover). Hence it is clear that the set

$$\bar{X}_1 = \bigcup_{\alpha} [W_\alpha]_{\bar{X}}$$

is closed in $\bar{X}$, i.e. the mapping $\bar{f}$ is perfect on $\bar{X}_1$. Since each set $[W_\alpha]_{\bar{X}}$ is contained in some $[Vx(y)]$, we have $\bar{X}_1 \subseteq O_1$. From the construction of the sets W_α it is also clear that the set

$$X_1 = \bigcup_{\alpha} W_\alpha$$

is open in X , i.e. on X_1 the mapping f is open and $f(X_1) = Y$. Replacing, at the second step, the space X , the space $\bar{X}$, and the set O_1 , respectively, by the set X_1 , the set $\bar{X}_1$, and the set $O'_2 = O_2 \cap \bar{X}_1$, we obtain a set $X_2 \subseteq O'_2$, closed in $\bar{X}_1$, perfectly mapped onto Y by means of $\bar{f}$, and a set $X_2 \subseteq \bar{X}_2$, open in X_1 , mapped by means of f onto the whole space Y . Continuing the process, we obtain sets $\bar{X}_i$, $i = 1, 2, \dots$, closed in $\bar{X}$, the intersection F of which is contained in and closed-

then in $\bigcap O_i = X$. Since each $\bar{X}_i$ was mapped perfectly onto Y by means of $\bar{f}$, and since $\bar{X}_{i-1} \subseteq \bar{X}_i$, the intersection F is nonempty and, by means of $\bar{f}$, which coincides on $F \subseteq X$ with f , is mapped perfectly onto Y .

The set F is the intersection not only of the sets $\bar{X}_i$, but also of the sets X_i (for one may assume that $[X_{i+1}]_X \subseteq X_i$), i.e., it has type G_δ in X .

Since the perfect image of a complete space is complete, the space Y is complete (for the set F is complete).

If now the space X is locally complete, then it is the open image of a complete space Z under a mapping g . In Z , by what has been proved, there will be found a closed set F , mapped perfectly onto Y by means of $f \cdot g$. Thus the space Y is complete and part a) of Corollary 10 is proved. The set $g(F) = X'$ is closed in X and is mapped perfectly onto Y . Theorem 8 is proved (for $Y' = Y$).

Note added in proof. The result of part a) of Corollary 10 was recently also obtained by Wicke.

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