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Abstract

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CYBERNETICS AND CONTROL THEORY

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ON THE PROBLEM OF A GAME ENCOUNTER OF MOTIONS

Consider the problem ⁽¹⁻⁵⁾ of the minimax of the time T until the encounter of the pursuing ($y(t)$) and pursued ($z(t)$) motions

$$\dot{y} = Ay + Bu, \quad \dot{z} = Az + Bv, \quad (1)$$

where the controls u and v are constrained by

$$\|u(t)\| \leq \mu, \quad \|v(t)\| \leq \nu \quad (\mu > \nu - \text{const}), \quad (2)$$

and the symbol $\|q\|$ denotes the Euclidean norm of the vector q . We shall treat the vectors under consideration as column vectors; the superscript $*$ will denote transposition; the symbol $[Q]_m$ will denote the matrix composed of the first m rows of the matrix Q . The objective of pursuit consists in the coincidence of the vectors $[y(t)]_m$ and $[z(t)]_m$. The control u must be formed according to the feedback principle at each instant $t = \tau$, on the basis of the realized values $y(\tau)$ and $z(\tau)$, i.e. $u[\tau] = u[y(\tau), z(\tau)]$. In order to distinguish the programmed controls u and v , specified a priori as functions of the time t , from the realizations of the controls u and v constructed according to the feedback principle,

$$u = u[y, z], \quad v = v[y, z], \quad (3)$$

but realized in concrete cases as functions of the time t , we shall denote the former by the symbols $u(t), v(t)$, and the latter by the symbols $u[t]$ and $v[t]$. We assume that the coordinates $y_j(t)$ and $z_j(t)$ ($j = 1, \dots, m$) are controllable, i.e. for any $y(\tau)$ and $z(\tau)$ and for any $\vartheta > \tau$ one can choose $u(t)$ and $v(t)$ ($\tau \leq t < \vartheta$) ensuring the equalities $[y(\vartheta)]_m = 0$, $[z(\vartheta)]_m = 0$. Thus we have the problem: find

$$T_*^0 = \min_u \sup_v T \quad \text{for} \quad [y(\tau + T)]_m = [z(\tau + T)]_m = 0, \quad (4)$$

Fig. 1

Figure 1: Fig. 1

where u is chosen among functions (3), and v among functions (3) and among controls $v(t)$. We shall call the solution $u^0[y(\tau), z(\tau)]$ of problem (4) the optimal control.

Problem (4), like other problems of optimal synthesis, is connected with suitable programmed problems. A rigorous justification of such an approach in the general case of a differential game, as is well known ⁽²⁾, encounters great difficulties. In the particular case under consideration, the desired minimum T^0 of $\max_v T$ is determined by the instant of absorption ^(4,5) ϑ^0 of the process $z(t)$ by the process $y(t)$, and the construction of the control u^0 can be connected with the rule of extremal aiming described in ^(4,5). The purpose of the present article is to discuss this rule in the case (1), (2). The extremal control $u[y(\tau), z(\tau)]^0$ has the form

$$u[y(\tau), z(\tau)]^0 = \mu h_{x(\tau)}^0(\tau + 0) / \|h_{x(\tau)}^0(\tau + 0)\|. \quad (5)$$

Here $x(\tau) = y(\tau) - z(\tau)$; $h_{x(\tau)}^0(t) = l^{0*}[F(\vartheta^0 - t)B]_m$ ($\tau \leq t < \vartheta^0$) — minimal function, moreover

$$\int_{\tau}^{\vartheta^0} \|l^{0*}[F(\vartheta^0 - t)B]_m\| dt = \min_l \int_{\tau}^{\vartheta^0} \|l^*[F(\vartheta^0 - t)B]_m\| dt = \kappa[x(\tau), \vartheta^0] = \frac{1}{\zeta} \quad (6)$$

$$(l^*[F(\vartheta^0 - \tau)x(\tau)]_m = -1),$$

where $\zeta = \mu - \nu$, ϑ^0 is the smallest root of the equation $\kappa[x(\tau), \vartheta] = 1/\zeta$, and $F(t)$ is the fundamental matrix of the equation $\dot{x} = Ax$.

By choosing $v = \nu u / \mu$ it is verified that, for any $u = u[y(\tau), z(\tau)]$, the inequality $\sup_v T \geq \vartheta^0 - \tau$ holds, and equality is possible only when $u = [y(\tau), z(\tau)]$ (in essence), i.e. $T^0 \geq \vartheta^0 - \tau$. The most convenient is the nondegenerate case in which $\|h_{x(\tau)}^0(\tau)\| > 0$ for all τ (for each function $h_{x(\tau)}^0(t)$). In this case, for $u = u[y(\tau), z(\tau)]^0$ and for the corresponding admissible v , the first equation (1) works as an ordinary differential equation; problem (4) has the solution $T^0 = \vartheta^0 - \tau$ and $v^0[y(\tau), z(\tau)] = u[y(\tau), z(\tau)]^0$.

Fig. 1

In the special case, the simple example

$$\dot{y}_1 = y_2, \quad \dot{y}_2 = -y_1 + u; \quad \dot{z}_1 = z_2, \quad \dot{z}_2 = -z_1 + v, \quad (7)$$

where it is required to achieve meeting in both coordinates, shows that the solution of problem (4) becomes seriously more complicated. In example (7), equality (5) means that the control $u[y(\tau), z(\tau)]^0$ is determined from the phase vector $x = y(\tau) - z(\tau)$ in the following way (see Fig. 1). Above the curve L we have $u[y(\tau), z(\tau)]^0 = -\mu$, below the curve L we have $u[y(\tau), z(\tau)]^0 = \mu$, and on the curve L : $u[y(\tau), z(\tau)]^0 = -\mu$ for $x_1 < 0$ and $u[y(\tau), z(\tau)]^0 = \mu$ for $x_1 > 0$. The function $v[y(\tau), z(\tau)]$ can be selected so that, for $u = u[y(\tau), z(\tau)]^0$, a sliding mode will arise on the curve L .^{*} This mode can be included among the admissible motions by introducing sequences $\{\tau_k\}$ ($\tau_{k+1} - \tau_k = \delta > 0$) and setting $u[\tau] = u[y(\tau_k), z(\tau_k)]$ for $\tau_k \leq \tau < \tau_{k+1}$, and then letting δ tend to zero. However, the parameters μ and ν and the function v can then be chosen so that, for $u = u[y(\tau), z(\tau)]^0$ and for every arbitrarily small $\delta > 0$, cyclic or quasicyclic motions will arise in a neighborhood of the point A . Therefore, as $\delta \rightarrow 0$, it is reasonable to treat this point as a point of rest on the sliding mode, which, consequently, does not converge at the point $x = y - z = 0$. The indicated circumstances lead to a modification of the problem. We introduce the above-mentioned sequences $\{\tau_k\}$ and admit among the arguments of the function u one more quantity ϑ , which will also change only at the instants $\tau = \tau_k$. In this case, the quantity ϑ at the instant τ_k must be determined only from the values $y[\tau_k]$, $z[\tau_k]$, and $\vartheta[\tau_{k-1}]$. (If pursuit began at $\tau > \tau_{k-1}$, then $\vartheta[\tau_k]$ is determined only from the values $y[\tau_k]$, $z[\tau_k]$.) We estimate the result of the pursuit by the quantity

$$\gamma_u = \sup_{\varepsilon} \left\{ \limsup_{\delta \rightarrow 0} \left[\sup_v T_{u,v}^{\varepsilon} \right] \right\},$$

where $\varepsilon > 0$, and $t = \tau + T_{u,v}^{\varepsilon}$ is the instant at which, for the chosen u and v , for the first time

$$\| [y(t)]_m - [z(t)]_m \| \leq \varepsilon.$$

The problem consists in choosing a control $u^0[y, z, \vartheta]$ that ensures the minimum

$$T^0 = \gamma_{u^0} = \min_u \gamma_u. \quad (8)$$

^{*} The author's attention was drawn to this circumstance, which served as the source of the present work, by Yu. M. Repin, who modeled similar modes on analog devices (see ⁽⁴⁾).

The following assertion is valid:

Problem (8) has a solution, and $T^0 = \vartheta^0 - \tau$. At the points $x = y(\tau) - z(\tau)$, where $\|h_x^0(\tau)\| > 0$ (for each function $h_x^0(t)$), on limiting motions we have $u^0[\tau] = u[y(\tau), z(\tau)]$, whatever admissible v may be. At the points x where $\|h_x^0(\tau)\| = 0$, the limiting values $u^0[\tau]$ (defined conditionally, as a regularization of the sliding regime) depend, generally speaking, on v .

The assertion is proved by constructing the control $u^0[y[t], z[t], \vartheta[t]]$ in a form analogous to (5). In this case, by introducing the quantity (8), regularization is achieved by replacing, at each step τ_k , the irregular problems of minimizing and maximizing the time of encounter ϑ by more regular problems of minimizing and maximizing the distance between the vectors $[y(\vartheta)]_m$ and $[z(\vartheta)]_m$, or the intensities of the controls $\text{vrai max } \|u(t)\|$, $\text{vrai max } \|v(t)\|$ ($\tau \leq t < \vartheta$) for fixed time ϑ . We emphasize that the regularization of the problem associated with the introduction of $\vartheta[\tau]$ and γ_u preserves the game condition: in computing $u[\tau]$, information about the choice of $v[t]$ or $v(t)$ ($t \geq \tau$) is not used.

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