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Abstract

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MATHEMATICS

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ON BEST MEAN APPROXIMATION BY POLYNOMIALS OF FUNCTIONS HAVING A REAL SINGULAR POINT

(Presented by Academician S. N. Bernstein on May 6, 1966)

S. N. Bernstein ⁽¹⁻³⁾ first observed that the sequence of best uniform approximations of the function $|x - c|^p$ ($-1 < c < 1$, $p > 0$) on the interval $[-1, 1]$ by algebraic polynomials of degree n is asymptotically equal to $(\sqrt{1 - c^2})^p \mu(p) n^{-p}$ ($n \rightarrow \infty$), where $\mu(p)$ is a certain constant, and indicated a method for finding the value of $\mu(p)$. These works play a fundamental role in the study of the properties of best approximation by polynomials of other functions with singularities of the same type.

Subsequently I. I. Ibragimov ⁽⁴⁾ applied S. N. Bernstein's method to the estimation of best uniform approximation of functions of the form $x^r |x|^\alpha \ln^m |x|$ ($r + \alpha > 0$, $m \geq 0$, r and m are integers, α is a real number) on the interval $[-1, 1]$ by means of a polynomial of degree n ; he also considered the case of the function $(a - x)^{r+\alpha} \ln^m(a - x)$ ($a \geq 1$).

S. M. Nikol'skii ⁽⁵⁾ investigated the asymptotic properties of the best approximation of the function $|a - x|^s$ by polynomials in the metric of the space L on the interval $[-1, 1]$, and obtained the corresponding asymptotic equalities.

The present note is devoted to best approximation by polynomials in the metric of the space $L_q(-1, 1)$ ($1 \leq q \leq \infty$) of the function $(x - a)^r \times |x - a|^\alpha \ln^m |x - a|$, where r and m are integers, and α and a are real numbers. In these investigations the main role is played by a general limit theorem of S. N. Bernstein (see ⁽⁶⁾, Theorem VII bis), and also Theorem 1, generalizing the well-known method of S. N. Bernstein ⁽³⁾ (this theorem is analogous to S. N. Bernstein's theorems not only in formulation, but also in the method of proof).

Theorem 1 (generalization of Bernstein's inequality). *Suppose that for the best approximation of a function $f(x)$ ($f(x) \in L_q(-2, 2)$) the following conditions are satisfied*

$$1) \quad E_{n-1}(f; a, b)_{L_q} < \left(1 + \frac{B \ln n}{\alpha_n}\right) E_n(f; a, b)_{L_q} \quad (-2 < a < 0 < b < 2) \quad (1)$$

where $B > 0$ is some constant; α_n is a monotonically increasing sequence ($1 \leq \alpha_n \leq n$) such that, for any constant γ ($0 \leq \gamma < 1$),

$$\lim_{n \rightarrow \infty} \frac{\ln n}{\alpha_n^{1-\gamma}} = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{\alpha_{n+o(n)}}{\alpha_n} = 1.$$

$$2) \quad E_n(x^{2p\nu} f(x); a, b)_{L_q} < \frac{C^{2\nu}(2\nu)^{2\nu}}{n^{2\nu}} E_n(f; a, b)_{L_q}, \quad (2)$$

where p and ν are natural numbers; $C > 0$ is some constant independent of ν and n .

3) There exists a natural number j such that

$$\lim_{n \rightarrow \infty} n^j E_n(f; a, b)_{L_q} = \infty. \quad (3)$$

Then

$$\begin{aligned} E_n \left(\sum_{k=1}^l B_k f(x - a_k); -1, 1 \right)_{L_q} &= \\ &= [1 + o(1)] \left\{ \sum_{k=1}^l |B_k|^q E_n^q(f(x - a_k); -1, 1)_{L_q} \right\}^{1/q}, \end{aligned} \quad (4)$$

where B_k are constants, $-1 < a_k < 1$, $a_k \neq a_i$ ($k \neq i$), $l < \infty$.

Consider the best weighted approximation

$$E_{n,r(x)}(f; a, b)_{L_q} = \inf_{c_k} \left\{ \int_a^b \left| f(x) - \sum_{k=1}^n c_k x^k \right|^q [r(x)]^q dx \right\}^{1/q}, \quad (5)$$

where the weight $r(x)$ satisfies the following conditions:

1) $r(x) \geq \beta > 0$;

2)

$$\left\{ \int_a^b [r(x)]^q dx \right\}^{1/q} \leq M < \infty \quad (-2 < a < 0 < b < 2);$$

3) the function $r(x)$ is continuous at the point $x = 0$.

Corollary. If the function $f(x)$ is continuous outside every neighborhood of zero and satisfies the conditions of Theorem 1, then the following asymptotic equality holds ($n \rightarrow \infty$):

$$E_{n,r(x)}(f; a, b)_{L_q} = [1 + o(1)]r(0)E_n(f; a, b)_{L_q}. \quad (6)$$

Lemma. For any $0 < \delta_n < 1$, for all natural $n > 1$, on intervals $[a, b]$, where $a < 0 < b$, the inequality holds ($r + \alpha > -1/q$, r and $m \geq 0$ are integers)

$$E_{n-1}(x^r |x|^\alpha \ln^m |x|; a, b)_{L_q} \leq \frac{(1 + \delta_n)^{n+1/q} + 1}{(1 + \delta_n)^{n-r-\alpha} - 1} \times \\ \times \left(1 + \frac{C(1 + \delta_n)^{n-r-\alpha} \ln(1 + \delta_n)}{[(1 + \delta_n)^{n+1/q} + 1] \ln n} \right) E_n(x^r |x|^\alpha \ln^m |x|; a, b)_{L_q}, \quad (7)$$

where $C > 0$ is some constant.

If in this inequality we put $(1 + \delta_n)^{n-r-\alpha} = 2n$, then we obtain (see (3, 7))

$$E_{n-1}(f; a, b)_{L_q} < \left(1 + \frac{D \ln n}{n} \right) E_n(f; a, b)_{L_q}, \quad (8)$$

where $D > 0$ is some constant.

Thus, for the function $x^r |x|^\alpha \ln^m |x|$ condition (1) of Theorem 1 is satisfied. It is not difficult to verify that for this function conditions (2) with $p = 1$ and condition (3) are satisfied. Thanks to the general limit theorem of S. N. Bernstein (6), this makes it possible to prove the following theorem.

Theorem 2. If $r + \alpha > -1/q$, $m \geq 0$, and $|a_k| < 1$ (r and m are integers, α and a_k are real numbers), then the equality holds ($n \rightarrow \infty$)

$$E_n \left(\sum_{k=1}^l B_k (x - a_k)^r |x - a_k|^\alpha \ln^m |x - a_k|; -1, 1 \right)_{L_q} = \\ = [1 + o(1)] \left\{ \sum_{k=1}^l \left| B_k \left(\sqrt{1 - a_k^2} \right)^{r+\alpha+1/q} \right|^q \right\}^{1/q} E_n(x^r |x|^\alpha \ln^m |x|; -1, 1)_{L_q} \quad (9)$$

(B_k are constants, $l < \infty$).

If α is not an even integer, then

$$\lim_{n \rightarrow \infty} \frac{n^{r+\alpha+1/q}}{(\ln n)^m} E_n(x^r |x|^\alpha \ln^m |x|; -1, 1)_{L_q} = A_1(x^r |x|^\alpha)_{L_q} < \infty; \quad (10)$$

if α is an even integer, then

$$\lim_{n \rightarrow \infty} \frac{n^{r+\alpha+1/q}}{(\ln n)^{m-1}} E_n(x^{r+\alpha} \ln^m |x|; -1, 1)_{L_q} = mA_1(x^{r+\alpha} \ln |x|)_{L_q} < \infty, \quad (11)$$

where $A_1[f(x)]_{L_q}$ is the best approximation of the function $f(x)$ by entire functions of degree ≤ 1 in the metric of the space $L_q(-\infty, \infty)$.

Relations (9) and (10), when r is even and $m = 0$, for $q = \infty$ constitute the theorem of S. N. Bernstein ⁽³⁾, and for $q = 1$ the result of S. M. Nikol'skii ⁽⁵⁾. For arbitrary natural r and $m = 0$, for $q = \infty$ this asymptotic equality was proved in the book of A. F. Timan ⁽⁷⁾, and for arbitrary q it was given in a note ⁽⁸⁾. Relations (10) and (11) were obtained by I. I. Ibragimov ⁽⁴⁾ ($q = \infty$) (see also ⁽⁹⁾). In the case $l = m = q = 1$, $r = \alpha = 0$, equalities (9) and (11) were obtained by another method by V. I. Gukevich ⁽¹⁰⁾.

Using the corollary of Theorem 1, we obtain the theorem:

Theorem 3. Whatever the noninteger $s > -1/2q$ and the integer $m \geq 0$ may be, for the best approximation of the function $(1-x)^s \ln^m(1-x)$ in the metric of the space L_q ($1 \leq q < \infty$) with weight $(1-x^2)^{-1/2q}$ by algebraic polynomials on $[-1, 1]$ there exists the limit

$$\lim_{n \rightarrow \infty} \frac{n^{2s+1/q}}{(\ln n)^m} E_{n, (1-x^2)^{-1/2q}}[(1-x)^s \ln^m(1-x); -1, 1]_{L_q} = 2^{m-s-1/q} A_1(|x|^{2s})_{L_q}, \quad (12)$$

and in the case when $s \geq 0$ and $m > 0$ are integers,

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{n^{2s+1/q}}{(\ln n)^{m-1}} E_{n, (1-x^2)^{-1/2q}}[(1-x)^s \ln^m(1-x); -1, 1]_{L_q} = \\ = m2^{m-s-1/q} A_1(x^{2s} \ln |x|)_{L_q}. \end{aligned} \quad (13)$$

For the best approximation of the function $(a-x)^s \ln^m(a-x)$ ($a > 1$) in the metric of the space $L_q(-1, 1)$, it is not difficult to obtain the inequalities

$$\frac{2^{(q+1)/q} (a^2 - 1)^{(s+1)/2} (\ln n)^m (1 - \varepsilon_n)}{|\Gamma(-s)| n^{s+1} (a + \sqrt{a^2 - 1})^{n+2}} \leq E_n[(a-x)^s \ln^m(a-x); -1, 1]_{L_q} \leq$$

$$\leq \frac{2^{1/q}(a^2-1)^{(s-1)/2}(\ln n)^m(1+\varepsilon_n)}{|\Gamma(-s)|n^{s+1}(a+\sqrt{a^2-1})^n}, \quad (14)$$

where $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$, s is not a nonnegative integer, and

$$\begin{aligned} \frac{2^{(q+1)/q}(a^2-1)^{(s+1)/2}s!m(\ln n)^{m-1}(1-\varepsilon_n)}{n^{s+1}(a+\sqrt{a^2-1})^{n+2}} &\leq E_n[(a-x)^s \ln^m(a-x); -1, 1]_{L_q} \leq \\ &\leq \frac{2^{1/q}(a^2-1)^{(s-1)/2}s!m(\ln n)^{m-1}(1+\varepsilon_n)}{n^{s+1}(a+\sqrt{a^2-1})^n}, \end{aligned} \quad (15)$$

if $s \geq 0$ is an integer.

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