

POLYHEDRALLY CLOSED SYSTEMS

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Abstract

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MATHEMATICS

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POLYHEDRALLY CLOSED SYSTEMS OF LINEAR INEQUALITIES OVER AN ARBITRARY ORDERED FIELD

(Presented by Academician V. M. Glushkov on 13 V 1966)

1. Let $L = L(P)$ be a linear space over an arbitrary ordered field P , and let $f_\alpha(x)$ ($\alpha \in M$) be linear (i.e., additive and homogeneous) functions defined on L , with values in P ; here M is some finite or infinite set of indices.

The system

$$f_\alpha(x) - a_\alpha \leq 0 \quad (\alpha \in M), \quad (1)$$

where $a_\alpha \in P$, will be called a **system of linear inequalities** on L . For system (1) with $a_\alpha = 0$, i.e., for a system of the form

$$f_\alpha(x) \leq 0 \quad (\alpha \in M) \quad (2)$$

we introduce the following definition. Let L' be some subspace of the space L^* of all linear functions (with values in P) defined on L , containing the elements f_α ($\alpha \in M$). System (2) will be called **polyhedrally** (L, L') -**closed** if its dual cone, i.e., the cone generated by the elements f_α ($\alpha \in M$) in L' , coincides with the intersection of all sets containing it that are defined in L' by inequalities of the form $x(f) \leq 0$ (x is a fixed element of L , and f is an element ranging over L').

Let $\bar{L}$ be the linear space of pairs $[x, t]$ ($x \in L$, $t \in P$), $\bar{L}'$ the linear space of pairs $[f, k]$ ($f \in L'$, $k \in P$), and

$$[f, k]([x, t]) = f(x) + kt.$$

System (1) will be called **polyhedrally** (L, L') -**closed** if the system

$$f_\alpha(x) - a_\alpha t \leq 0 \quad (\alpha \in M), \quad -t \leq 0 \quad (3)$$

is polyhedrally $(\bar{L}, \bar{L}')$ -closed. A polyhedrally (L, L^*) -closed system (1) ($L' = L^*$) will be called **polyhedrally closed**. In the author's paper ⁽¹⁾ it was noted that the system

$$f_\alpha(x) - a_\alpha = a_{\alpha 1}x_1 + \dots + a_{\alpha n}x_n - a_\alpha \leq 0 \quad (\alpha \in M) \quad (4)$$

over the space R^n (R is the field of real numbers) is polyhedrally closed if and only if its associated cone (the cone generated by the $(n+1)$ -dimensional vectors $(a_{\alpha 1}, \dots, a_{\alpha n}, -a_\alpha)$ ($\alpha \in M$) and $(0, \dots, 0, -1)$) is topologically closed in the space R^{n+1} .

In ⁽¹⁾ polyhedrally (L, L') -closed systems were considered in the case where the ground field P coincided with the field R ; however, in fact, the results presented there that concern the space $L(R)$ of general form (propositions (*), (**), Theorems 1 and 3, and Corollary 1 of Theorem 3) are valid not only for $P = R$, but also for an arbitrary ordered field P . We note here the following generalization, following from proposition (**), of the Alexandrov-Fan Ji theorem.

Theorem 1. *A polyhedrally (L, L') -closed system (1) (over $L = L(P)$) is consistent if and only if every identically*

equal to zero on the L -linear combination

$$p_{a_1}f_{a_1}(x) + \dots + p_{a_s}f_{a_s}(x)$$

(s is not fixed, $s \geq 1$) with positive coefficients from P , spanning a system of functions $f_{a_1}(x), \dots, f_{a_s}(x)$ of rank $s-1$, there corresponds the inequality

$$p_{a_1}a_{a_1} + \dots + p_{a_s}a_{a_s} \geq 0,$$

or when no such identically zero combination exists.

2. Let U be some subspace of L . The **cone** of the U -hull of the system of linear inequalities

$$f_a(x) + t_a \leq 0 \quad (a \in M) \quad (5)$$

over $L = L(P)$, in which t_a are parameters taking one or another value from the field P , will be called the set $C(U)$ of nonnegative solutions $\{u_a\}$ ($a \in M$) of the equation

$$\sum_{a \in M} u_a f_a(x) = 0 \quad (x \in U)$$

with a finite number of nonzero coordinates u_a . An element $\{u_a^0\}$ ($a \in M$) of the cone $C(U)$ will be called **fundamental** if the rank of the system of functions $f_a(x)$, whose indices a coincide with the indices of its nonzero coordinates u_a^0 , is one less than their number. The following assertion is valid:

(*)

Maximal systems of essentially distinct fundamental elements of a nonzero cone $C(U)$, and only they, are its bases.

Elements of the cone $C(U)$ that differ only by positive factors (from P) are not considered essentially distinct.

If

$$C^\beta = \{C_a^\beta\} \quad (a \in M, \beta \in N)$$

is some system of generating elements of the cone $C(U)$, then the system

$$\sum_{a \in M} C_a^\beta f_a(x) + \sum_{a \in M} C_a^\beta t_a \leq 0 \quad (\beta \in N) \quad (6)$$

will be called the U -**hull** of system (5) (for $t_\alpha = -a_\alpha$, the U -hull of system (1)). If C^β ($\beta \in N$) is a maximal system of essentially distinct fundamental elements of the cone $C(U)$, then we shall call the U -hull a **fundamental U -hull**. If the cone $C(U)$ is zero, then we shall say that the U -hull is **empty**. For $U = L$ we shall call the U -hull **complete**.

Using assertion (*), it is not difficult to verify that every U -hull contains a fundamental U -hull and may differ from the latter only by inequalities that are linear combinations with positive coefficients of its inequalities.

Theorem 2. System (5) with a nonempty U -hull for some $U \subseteq L$ is compatible for those values of the parameters t_a for which it is polyhedrally (L, L') -closed and has a compatible U -hull. System (5) with an empty U -hull is compatible for arbitrary values of the parameters t_a for which it is polyhedrally (L, L') -closed.

In essence, this theorem is equivalent to the assertion that a polyhedrally (L, L') -closed system (1) with a compatible or empty U -hull is compatible.

Corollary 1. The system (4) with a conjugate cone topologically closed in the space R^{n+1} is consistent if at least one of its U -convolutions is consistent or empty for some subspace $U \subseteq L = R^n$ (for example, at least one convolution).

Theorem 3. If U is some subspace of L and V^* is the subspace of all elements $f \in L^*$ for which $f(x) = 0$ ($x \in U$), then the dual cone of any U -convolution of the system (2) coincides with the intersection of the dual cone of the system (2) with the subspace V^* , if the U -convolution is nonempty, and with the zero element of L^* , if it is empty.

Corollary 1. If U is any subspace of L with respect to which the system (1) has a nonempty U -convolution, then from the polyhedral closedness of the system (1) there follows the polyhedral closedness of each of its U -convolutions.

Corollary 2. If the conjugate cone of the system (4) over the space R^n is topologically closed in the space R^{n+1} , then each nonempty U -convolution of it for any $U \subseteq R^n$ is polyhedrally closed.

Corollary 3. For any two subspaces U_1 and U_2 of L , the U_2 -convolution of the U_1 -convolution of the system (5) (of the system (1)) coincides with the $U_1 + U_2$ -convolution of the latter.

3. If there exists a least upper bound T ($T \in P$) of the values of the linear function $f(x)$ on the set H of solutions of the consistent system (1), then we shall say that T is the **greatest value** of the function $f(x)$ on the set H . Obviously, T is the least upper bound of the values of the parameter t for which the system

$$f_\alpha(x) - a_\alpha \leq 0 \quad (\alpha \in M), \quad -f(x) + t \leq 0 \quad (7)$$

is consistent.

Theorem 4. If a linear function $f(x)$ ($x \in L$, $f \in L'$) has a greatest value T on the set H of solutions of a consistent polyhedrally (L, L') -closed system (1) of nonzero rank, then the system (1) has such a finite subsystem S of rank equal to the number of its inequalities, on the set of solutions of which the greatest value of the function $f(x)$ exists, is attained, and coincides with T . If the subsystem S has no subsystems distinct from it of this kind and the value T is attained on the set H , then it is attained for those and only those solutions of the system (1) which satisfy the boundary equations of all inequalities of the subsystem S .

Theorem 5. In order that a nonzero linear function $f(x)$ ($x \in L$, $f \in L'$), having a greatest value T on the set H of solutions of a polyhedrally (L, L') -closed consistent system (1) of nonzero rank, attain it on H , it is sufficient that the corresponding system (7) with $t = T$ be polyhedrally (L, L') -closed.

The condition of the theorem is not necessary.

Example. The system

$$-\frac{1}{n(n+1)}x - y + \frac{2n+1}{n(n+1)}z \leq 0 \quad (n = 1, 2, \dots),$$

$$-y \leq 0, \quad -z \leq 0$$

is polyhedrally closed (here $L = L' = R^3$). The greatest value T of the function $-y$ on the set of its solutions is, obviously, equal to zero. It is attained for each

of its solutions $(x, 0, 0)$ with $x \geq 0$. The system (7) with $t = T$ here has the form

$$-\frac{1}{n(n+1)}x - y + \frac{2n+1}{n(n+1)}z \leq 0 \quad (n = 1, 2, \dots),$$

$$-y \leq 0, \quad y \leq 0, \quad -z \leq 0.$$

In view of Theorem 1 from the paper [1], it is not polyhedrally closed, since the inequality $z \leq 0$, which is its consequence, obviously,

cannot be represented as a linear combination with nonnegative coefficients of a finite number of its inequalities.

4. Denote by $M(P)$ the space that is the direct sum of the spaces P_α ($\alpha \in M$), isomorphic to the space P^1 . An arbitrary element x of $M(P)$ is a system $(\dots, x_\alpha, \dots)$ with a finite set of components $x_\alpha \in P_\alpha$ ($\alpha \in M$) different from zero, containing one and only one element x_α from each P_α . If c_α ($\alpha \in M$) is an arbitrary element of the (ordered) field P , then the expression $\sum_{\alpha \in M} c_\alpha x_\alpha$ will be called a **linear form** on the space $M(P)$. An element $x = (\dots, x_\alpha, \dots)$ of the space $M(P)$ will be called **nonnegative** if each of its components x_α is nonnegative, and **positive** if, in addition, at least one of them is positive.

The problem of minimizing the linear form $a(x) = \sum_{\alpha \in M} a_\alpha x_\alpha$ on the set N of nonnegative elements $u = (\dots, u_\alpha, \dots)$ of $M(P)$, for each of which the relation, identical with respect to $x \in L = L(P)$,

$$\sum_{\alpha \in M} u_\alpha f_\alpha(x) = f(x), \quad (8)$$

holds, will be called the **dual problem** for the problem of maximizing the linear function $f(x)$ ($x \in L$, $f \in L'$) on the set of solutions of the consistent polyhedrally (L, L') -closed system (1).

Theorem 6. *If there exists a greatest value T of the linear function $f(x)$ ($x \in L$, $f \in L'$) on the set H of solutions of a polyhedrally (L, L') -closed consistent system (1), then the set N is nonempty and on it there exists and is attained the least value of the linear form*

$$a(u) = \sum_{\alpha \in M} a_\alpha u_\alpha$$

and it coincides with T . Conversely, if the set N is nonempty and on it there exists the least value T of the linear form $a(u)$, then there exists the greatest value of the function $f(x)$ on the set H , and it coincides with T .

Thus, the linear function $f(x)$ ($x \in L$, $f \in L'$) has a greatest value on the set of solutions of the polyhedrally (L, L') -closed consistent system (1) if and only if the set N of nonnegative solutions $(\dots, u_\alpha, \dots)$ of equation (8), having a finite number of nonzero coordinates, is nonempty and on it there exists the least value of the linear form $a(u)$.

Theorem 7. *The linear function $f(x)$ ($x \in L$, $f \in L'$) has a greatest value on the set H of solutions of a polyhedrally (L, L') -closed system (1) and attains it on this set if and only if the system*

$$f_\alpha(x) - a_\alpha \leq 0 \quad (\alpha \in M), \quad \sum_{\alpha \in M} u_\alpha f_\alpha(t) = f(t) \quad (t \in L),$$

$$\sum_{\alpha \in M} a_\alpha u_\alpha \leq f(x), \quad -u_\alpha \leq 0$$

has at least one solution (x, u) , where $x \in H$, and u is a nonnegative vector $(\dots, u_\alpha, \dots)$ with a finite number of positive coordinates.

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REFERENCES

1. S. N. Chernikov, *DAN*, **161**, No. 1, 55 (1965).

Note: Figure translations are in progress. See original paper for figures.

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