

SYMMETRIC ALGEBRAS AND STEINER SYSTEMS

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1967

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Abstract

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UDC 519.123

MATHEMATICS

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SYMMETRIC ALGEBRAS AND STEINER SYSTEMS

(Presented by Academician A. I. Mal'cev, 15 VII 1966)

In this paper certain combinatorial designs are considered, i.e., combinations of elements of a given set possessing specified properties, and an algebraic interpretation of these designs is given. Such designs are Steiner triples and quadruples—they correspond to TS -quasigroups ⁽¹⁾ and S -algebras ⁽²⁾. On the set of finite S -algebras a binary operation is defined, distinct from the direct product.

Let a Steiner triple system M be given on a set A , i.e., a system of triples (x, y, z) of pairwise distinct elements of A , such that any pair $x, y \in A$ ($x \neq y$) occurs in one and only one triple of M . Adjoin to the system M the trivial triples (x, x, x) ; then we obtain the system $\bar{M}$. Define multiplication on A by putting $xy = z \iff (x, y, z) \in \bar{M}$. Then A becomes a TS -quasigroup ⁽¹⁾, i.e., a quasigroup with the identities

$$x^2 = x, \quad (xy)y = x, \quad x(xy) = y.$$

Obviously the converse is also true: if A is a TS -quasigroup, then on the set A one can define a Steiner triple system by putting $(x, y, z) \in M \iff xy = z$.

Adjoin to the TS -quasigroup A an element e and define on the set $\bar{A} = A \cup \{e\}$ a new operation $(\cdot)$, putting, for arbitrary $x, y \in A$ ($x \neq y$), $x \cdot y = xy$, $x \cdot e = e \cdot x = x$, $x \cdot x = e$, $e \cdot e = e$. Then $\bar{A}$ becomes a TS -loop with identity e , i.e., a loop with the identities $x^2 = e$, $(xy)y = x$, $x(xy) = y$. The correspondence obtained between TS -quasigroups and TS -loops is, obviously, invertible; thus the following assertion is true.

Proposition 1. *There exists a one-to-one correspondence between the sets: I—Steiner triple systems of order n ; II— TS -quasigroups of order n ; III— TS -loops of order $n + 1$.*

Let a Steiner quadruple system N be given on a set B , i.e., a system of quadruples (x, y, z, t) of pairwise distinct elements of B , such that any triple $x, y, z \in A$ ($x \neq y \neq z \neq x$) occurs in one and only one quadruple of N . Adjoin to the

system all possible quadruples of the form (x, x, y, y) ; then we obtain the system $\overline{N}$.

Define on B a ternary operation $xyzt$, putting

$$xyzt = t \iff (x, y, z, t) \in \overline{N}.$$

With respect to this operation B is an S -algebra in the sense of ⁽²⁾, i.e., an algebra with one basic ternary operation $xyzt$, satisfying the identities

$$xxyt = y, \quad xyzyt = x, \quad xxyztzt = y, \quad xyxyzt\tau = z. \quad (1)$$

We note that the identities (1) may also be written in the form of conditional identities

$$x = y \Rightarrow xyt = z, \quad xyt = t \Rightarrow tyt = x \ \& \ xtzt = y \ \& \ xytt = z$$

and imply the commutativity of the basic operation

$$xyzt = xzyt = zxy\tau.$$

It is easy to see that to each S -algebra B there corresponds a certain Steiner system of quadruples on the set B . Thus, the following assertion is true.

Proposition 2. *There exists a one-to-one correspondence between the set of Steiner systems of quadruples of order n and the set of S -algebras of order n .*

The class of S -algebras is determined by the identities (1); consequently, this class is a variety and, in particular, is closed under direct (Cartesian) products. On this class we shall define another operation, which we shall call the **product of S -algebras with an identified element** (or the matrix product). The construction of this product will be set out for the case of finite S -algebras, but it is applicable without any changes also to infinite S -algebras.

Let A be an S -algebra of order $t_1 + 1$, and B an S -algebra of order $t_2 + 1$. Choose one element in each of A and B and identify them; denote this common element by ε . Denote the remaining elements of A by a_i , and the elements of B by b_i ($1 \leq i \leq t_1$, $1 \leq j \leq t_2$). Let

$$C = \{\varepsilon\} \cup \{c_{ij}, 1 \leq i \leq t_1, 1 \leq j \leq t_2\}, \quad \bar{C} = \|c_{ij}\|.$$

On the set C we shall define a Steiner system of quadruples P , with which the construction of an S -algebra on C is associated; the resulting S -algebra of order $t_1 t_2 + 1$ will be the desired one. The construction of the system P is divided into three stages.

I. $t_1 = t_2 = 3$.

a) The quadruples from P containing ε will be the quadruples of the form

$$(\varepsilon, c_{i1}, c_{i2}, c_{i3}), \quad (\varepsilon, c_{1j}, c_{2j}, c_{3j}) \quad (i, j = 1, 2, 3),$$

as well as quadruples of the form

$$(\varepsilon, c_{i_1 j_1}, c_{i_2 j_2}, c_{i_3 j_3}),$$

where $(i_1, i_2, i_3), (j_1, j_2, j_3)$ are permutations of the numbers 1, 2, 3; the number of such quadruples is 12.

b) Combine into one quadruple the elements of any minor of order 2 of the matrix $\bar{C}$; then we obtain another 9 quadruples.

c) To each element c_{ij} assign the quadruple of elements located in the i -th row and in the j -th column of the matrix $\bar{C}$ and distinct from c_{ij} ; this rule gives another 9 quadruples.

Thus, 30 quadruples are obtained, and they form the complete system P ; we shall denote this system below by P_3 .

Remark. Every Steiner system of quadruples of order 10 is isomorphic to P_3 .

II. $t_1 > 3, t_2 = t > 3$.

a) Let (j_1, j_2, j_3) be an arbitrary triple of numbers for which

$$b_{j_1} b_{j_2} b_{j_3} \tau = \varepsilon.$$

Then consider the minor of order (3×3) situated in the columns of the matrix $\bar{C}$ with numbers j_1, j_2, j_3 . Associate with this minor (and with the element ε) the system P_3 ; then we obtain a certain set of quadruples, in particular all quadruples from P containing ε .

b) Let (j_1, j_2, j_3, j_4) be a quadruple of numbers for which

$$b_{j_1} b_{j_2} b_{j_3} \tau = b_{j_4}.$$

Then consider the minor of order (3×4) situated in the columns of the matrix $\bar{C}$ with numbers j_1, j_2, j_3, j_4 . Since the numbers j_k ($k = 1, 2, 3, 4$) are distinct, we may, without loss of generality, regard them as ordered increasingly:

$$j_1 < j_2 < j_3 < j_4.$$

From the elements of the indicated minor we construct quadruples of the form

$$(c_{i_1 j_1}, c_{i_2 j_2}, c_{i_3 j_3}, c_{i_4 j_4}),$$

where

$$(i_1, i_2, i_3, i_4) = (i, j, j, i), \quad (i, j, i, j) \quad (i, j = 1, 2, 3);$$

$$(i, i, j, k), \quad (i, j, k, k) = (j_1, j_2, j_3, j_4),$$

where (i, j, k) is a permutation of the numbers 1, 2, 3.

Rules a), b) give the complete Steiner system of quadruples over C ; denote it by P_t . Analogously, the Steiner quadruples are constructed in the case $t_1 = t > 3$, $t_2 = 3$; we shall denote this system by P'_t ($P_3 = P'_3$).

Remark. The construction of the system $P_t(P'_t)$ depends on the structure of the algebra $B(A)$, and therefore one ought to write $P_t(B)$, $P'_t(A)$. We use the abbreviated notation, since from the context it is clear which algebras are meant.

III. $t_1, t_3 > 3$.

a) Let (i_1, i_2, i_3) be an arbitrary triple of numbers for which

$$a_{i_1} a_{i_2} a_{i_3} \tau = \varepsilon.$$

Consider the minor of order $3 \times t_2$, situated in the rows

the matrix $\bar{C}$ with numbers i_1, i_2, i_3 . Associate with this minor (and with the element ε) the Steiner system P_{t_2} . Similarly, if (j_1, j_2, j_3) is a triple of numbers for which $b_{j_1} b_{j_2} b_{j_3} t = \varepsilon$, associate with the minor of the matrix $\bar{C}$ situated in the columns with numbers j_1, j_2, j_3 the system of quadruples P'_{t_1} .

b) Let $(i_1, i_2, i_3, i_4), (j_1, j_2, j_3, j_4)$ be quadruples of pairwise distinct numbers for which

$$a_{i_1} a_{i_2} a_{i_3} \tau = a_{i_4}, \quad b_{j_1} b_{j_2} b_{j_3} \tau = b_{j_4}.$$

Then put

$$(c_{i_1 j_1}, c_{i_2 j_2}, c_{i_3 j_3}, c_{i_4 j_4}) \in P.$$

Rules a), b) exhaust the construction of the Steiner system of quadruples. In each of the cases I-III, denote by $A \circ B$ the S -algebra corresponding to the constructed system of quadruples. We summarize the results of the construction described above in the following theorem.

Theorem. Let A_i ($i = 1, 2$) be an S -algebra of order $t_i + 1$, $t_i \geq 3$. Then there exists an S -algebra of order $t_1 t_2 + 1$, containing t_1 subalgebras isomorphic to A_2 , t_2 subalgebras isomorphic to A_1 , $t_1(t_1 - 1)/6$ subalgebras isomorphic to P_{t_2} , $t_2(t_2 - 1)/6$ subalgebras isomorphic to P'_{t_1} , and

$$t_1 t_2 (t_1 - 1)(t_2 - 1)/36$$

subalgebras of order 10.

To prove the theorem it is enough to consider the product $A_1 \circ A_2$ of the S -algebras A_1, A_2 with identified element. Note that the S -algebras $A_1 \circ A_2$ and $A_2 \circ A_1$ are isomorphic. However, the question of whether the product $A_1 \circ A_2$ depends on the choice of the elements in A_1, A_2 subject to identification remains open. In this connection, the question of whether the TS -loops obtained from a given S -algebra A by introducing the derived operation $x \cdot y = xy\alpha$, as the element varies, will be isomorphic to one another is also of interest.

The author expresses his gratitude to E. N. Kuzmin for discussion of the results.

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Received

11 VII 1966

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Note: Figure translations are in progress. See original paper for figures.

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