

UNATTAINABLE SUBGROUPS AND THE NORMAL STRUCTURE OF FINITE GROUPS

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Abstract

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MATHEMATICS

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UNATTAINABLE SUBGROUPS AND THE NORMAL STRUCTURE OF FINITE GROUPS

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§ 1. The presence in a group of unattainable subgroups and their mutual arrangement may exert a very substantial influence on the normal structure of the group. An example of this is the well-known theorem of Frobenius ⁽¹⁾, according to which the existence in a finite group of a subgroup that coincides with its normalizer and is mutually prime with all its conjugate subgroups, i.e. is “very far” from a normal divisor, entails the existence in the group of an additional invariant subgroup. Moreover, as J. Thompson ⁽²⁾ proved, the additional invariant set is always nilpotent.

In § 3 finite groups with unattainable subgroups are considered and, under certain additional requirements, the following are proved: the existence of an invariant complement in the group (Theorems 3 and 4), and solvability of the group (Theorems 1, 2, 5, and 6). Theorem 7 gives a description of groups for which $\text{ev}(G) = 4$ (see the definition below).

In § 4 the normal structure of a finite group is investigated in connection with the permutability of its subgroups.

Since groups in which all i -th ($i = 2, 3$) maximal subgroups are invariant are not generated by their i -th ($i = 2, 3$) maximal unattainable subgroups, Theorems 23 and 24 of B. Huppert ⁽⁴⁾, respectively, are special cases of Theorems 1 and 2; Theorem 2 generalizes Theorem 12 of Ya. G. Berkovich ⁽⁵⁾; Theorem 3 is an analogue of a theorem of R. Carter ⁽⁶⁾; Theorem 5 generalizes Herstein's theorem ⁽⁷⁾ on the solvability of groups with a maximal abelian subgroup; from Theorem 7 follows the main result of V. Deskins ⁽³⁾; the corollary generalizes the known theorem of B. Huppert ⁽⁸⁾ on the supersolvability of the product of two permutable cyclic groups; Theorem 9 refines Theorem 5 of B. Huppert ⁽⁹⁾; Theorems 10 and 11 supplement the estimates of Hall–Higman ⁽¹⁰⁾ for the p -length of a p -solvable group (Theorem 1.26 and Theorem A).

§ 2. We shall use the following definitions and notation: G is a finite group of order (G) ; Π is some set of prime numbers; $(G)_{\Pi}$ is the greatest Π -divisor of (G) ; an S_{Π} -subgroup is a subgroup of order $(G)_{\Pi}$ of the group G ; $\lambda(m)$ is the number of all, not necessarily distinct, prime divisors of the natural number m .

A group G is called **Π -decomposable** if it decomposes into a direct product of two sets, one of which is its nilpotent S_Π -subgroup ⁽¹¹⁾.

Let P be some Sylow p -subgroup of the group G , and let

$$z_0 = 1 \subseteq z_1(P) \subseteq z_2(P) \subseteq \dots \subseteq z_n(P) = P$$

be the upper central series of the subgroup P .

A group G is called **strongly p -normal** if $z_i(P)$, $i = 1, 2, \dots, n-1$, is invariant in every Sylow p -subgroup containing it ⁽¹²⁾.

A finite group G is called, according to S. A. Chunikhin ⁽¹³⁾, p -solvable (p -supersolvable) if G has a composition (chief-

row), each index of which is either equal to the prime number p , or is not divisible by p .

Let

$$1 = P_0 \subseteq N_0 \subset P_1 \subset N_1 \subset \dots \subset N_l = G$$

be the upper invariant p -series of the p -solvable group G , i.e., such a series in which N_i/P_i ($i = 0, 1, \dots, l$) is the largest invariant p' -subgroup in G/P_i , and P_i/N_{i-1} is the largest invariant p -subgroup in G/N_{i-1} ($i = 1, 2, \dots, l$).

The minimal number l for which $N_l = G$ is called, following Hall and Higman ⁽¹⁰⁾, the p -length of the p -solvable group G .

A set of subgroups of a finite group G , $M_0 = G, M_1, \dots, M_n$, is called an upper even chain C_n of length n if each subgroup M_i is maximal in M_{i-1} , $i = 1, 2, \dots, n$, and has even order or is the identity subgroup.

Following V. Deskins ⁽³⁾, we introduce the concept of even variation for a group G . The ratio $n/s(C_n)$ ($s(C_n)$ is the number of subgroups $M_i \neq M_0$ of the upper even chain C_n that are attainable in G), if $s(C)_n \neq 0$, and n if $s(C_n) = 0$, is called the even variation of C_n . Among the even maximal chains of subgroups of the group G we choose one that has the greatest even variation; the variation of this chain is called the even variation of the group G and is denoted by $\text{ev}(G)$.

On Schmidt groups, see ^(14,15). On the groups $SL(2, 5)$ and $LF(2, p^n)$, see ⁽¹⁶⁾ or ⁽¹⁷⁾.

We present the results obtained.

§ 3. Theorem 1. *Let G not be generated by a set of second maximal unattainable subgroups. Then G is solvable. If G is a non-nilpotent group, then it can be of two types:*

- 1) $G = PQ$, where P is a cyclic p -group and Q is a minimal normal divisor of the group G .

- 2) $G = (PQ) \times R$, where P and R are cyclic subgroups, and PQ is a Schmidt group, with Q a minimal normal divisor of the group G .

Theorem 2. *If G is not generated by a set of third maximal unattainable subgroups, then G is solvable.*

Theorem 3. *Let H be a nilpotent S_Π -subgroup that coincides with its normalizer in the group G . If G is strongly p -normal for all $p \in \Pi$, then H has an invariant complement in G .*

Theorem 4. *Let H be an S_Π -subgroup of the group G , and suppose that some maximal subgroup M of G containing H is Π -decomposable. If H has a Sylow p -subgroup whose center is not invariant in G , and if G is strongly p -normal, then G has an invariant complement to H .*

Theorem 5. *Let M be a maximal nilpotent subgroup of the finite group G . If G is strongly 2-normal, then it is solvable and its 2-length is equal to 1.*

Theorem 6. *A finite group of even order is solvable if $\text{ev}(G) < 4$.*

Theorem 7. *Let G be a nonsolvable group with $\text{ev}(G) = 4$. Then G is isomorphic to one of the following groups:*

- 1) the special linear group $SL(2, 5)$;
- 2) the fractional-linear group $LF(2, p^n)$, except that: a) if 4 divides $(p - 1)$, then $\lambda(p - 1) \leq 3$, $n = 1$; b) if 4 divides $(p + 1)$, then $\lambda(p^n + 1) \leq 3$, $p \neq 7$, $\lambda(p^n - 1) \leq 3$.

Theorem 6 weakens the condition of theorem 3 of ⁽³⁾ for groups of even order, and theorem 7 weakens the condition of theorem 17 of ⁽¹⁸⁾.

§ 4. **Theorem 8.** *Let the p -solvable group G be representable in the form $G = AB$, where A and B are p -decomposable groups with cyclic Sylow p -subgroups. Then G is p -supersolvable.*

In the theorem just given, the condition of p -decomposability of the factors is essential, as is seen from the example of the octahedral group, which is not 2-supersolvable, although it is representable as a product of subgroups (6.4) with cyclic Sylow p -subgroups.

Since the product of permutable nilpotent subgroups is soluble ⁽¹⁹⁾, Theorem 8 implies

Corollary. *Let $G = AB$, where A and B are nilpotent groups, and let the Sylow p -subgroups of A and B be cyclic. Then G is p -supersolvable.*

If the condition of the corollary is satisfied for all prime divisors of the order of the group, then we obtain a result of B. Huppert: a finite group admitting a factorization by two cyclic subgroups is supersolvable ⁽⁸⁾.

Theorem 9. *Let P be a Sylow p -subgroup and let Q be a p -complement in G ; suppose that $(P) > p$ and every maximal subgroup of P is permutable with Q .*

Then G is a p -supersolvable group.

Let P be a Sylow p -subgroup of the group G , and let

$$P = \Gamma_0(P) \supseteq \Gamma_1(P) \supseteq \cdots \supseteq \Gamma_n(P) = 1$$

be its lower central series.

Theorem 10. Let G be a p -solvable group with Sylow p -subgroup P and p -complement Q . If the k -th term ($1 \leq k \leq n$) of the lower central series of the Sylow p -subgroup P is permutable with Q , then $l_p(G) \leq k$.

From Theorem 10, when $k = 1$, there follows one result of ⁽²⁰⁾.

Theorem 11. Let G be a p -solvable group with Sylow p -subgroup P and p -complement Q . If the k -th term ($1 \leq k \leq n$) of the commutator series of the Sylow p -subgroup P is permutable with Q and $p > 2$, then $l_p(G) \leq k$.

Theorem 12. Let G be a p -solvable group with Sylow p -subgroup P and p -complement Q . If the first k , distinct from E , terms of the upper central series of the Sylow p -subgroup P are permutable with G , then $l_p(G) \leq c_p - k$.

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