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Abstract

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UNIQUENESS CLASSES FOR THE SOLUTION OF THE CAUCHY PROBLEM

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For systems of linear partial differential equations with constant coefficients, uniqueness classes for the solution of the Cauchy problem were obtained in ^(1,2). The question of the sharpness of these classes was subsequently studied in ^(3,4).

In the present paper we obtain conditions (as a rule, necessary and sufficient) for the uniqueness of the solution of the Cauchy problem for equations of the form

$$\frac{\partial^m u(x, t)}{\partial t^m} = \sum_{k=0}^n q_k(x) \frac{\partial^k u(x, t)}{\partial x^k}, \quad (1)$$

$$-\infty < x < \infty; \quad t \geq 0; \quad q_k(x), \quad k = 0, 1, \dots, n,$$

are complex-valued functions; $q_n(x) \equiv a = \text{const} \neq 0$, with initial conditions

$$D_t^j u(x, 0) = \varphi_j(x), \quad j = 0, 1, \dots, m-1. \quad (1')$$

In addition, necessary conditions for uniqueness of the solution of the Cauchy problem for systems of equations of the form (1) are obtained. Here the conditions on the coefficients of equation (1) are such that, generally speaking, arbitrary growth as $|x| \rightarrow \infty$ is allowed in the case $n \leq m$, while in the case $n > m$ "slow" growth of the coefficients is allowed. More rapid growth of the coefficients $q_k(x)$ for $n > m$ leads to substantial changes in the uniqueness classes; these questions are the subject of our following note ⁽⁸⁾. However, for second-order equations ($n = 2, m = 1$), exact uniqueness classes for the solution of the Cauchy problem are indicated in our note ⁽⁵⁾.

Theorem 1. *Suppose that the following conditions are satisfied:*

1. The functions $q_k(x)$, $k = 0, 1, \dots, n-1$, have continuous derivatives, and the estimates

$$|q_k(x)| \leq [h(x)]^{n-k}, \quad -\infty < x < \infty, \quad (2)$$

hold, where $h(x) > 0$ is some even monotone function for which

$$\int_{-\infty}^{\infty} [h(x)]^{-n/m} dx = \infty. \quad (3)$$

2. For sufficiently large values of $|x|$,

$$\sup_{|t| \leq |x|} |q'_k(t)| \leq \frac{C|q_k(x)|}{1 + |x|}, \quad k = 0, 1, \dots, n-1. \quad (4)$$

3. $\text{Im}(ai^m) \neq 0$; $\text{Im } a \neq 0$ when $m = 2m_1 + 1$, $n = 2n_1 + 1$; $\text{Re } a \neq 0$ when $m = 2m_1$, $n = 2n_1 + 1$.

Then the Cauchy problem (1)–(1') has a unique solution in the class of functions

$$|D_x^j f(x, t)| \leq C_f \exp \left\{ at + \left| \int_0^x H(t) dt \right| \right\}, \quad 0 \leq j \leq n-1, \quad (5)$$

where $a > 0$ is any fixed number; $H(t) > 0$ is an even monotone function satisfying the condition

$$\int_{-\infty}^{\infty} [H(x)]^{1-n/m} dx = \infty. \quad (6)$$

If condition 3 is not fulfilled, then the same result is valid if, in conditions (3) and (6), the exponent $1 - n/m$ is replaced by $1 - n/m - \varepsilon$ for any (fixed) $\varepsilon > 0$.

Remark. For $n = m$ and when condition 3 is fulfilled, and also for $n < m$, independently of whether condition 3 is fulfilled, we obtain that uniqueness of the solution of the Cauchy problem (1)–(1') holds in the class of all functions without restrictions on their growth as $|x| \rightarrow \infty$, for arbitrarily increasing coefficients $q_k(x)$. (In these cases condition 2 is not used.)

If $n = m$ and condition 3 is not fulfilled, then condition (2) allows arbitrary power growth of the coefficients, and in this case uniqueness of the solution of the Cauchy problem (1)–(1') holds in any class of functions of the form $|D_x^j f(x, t)| \leq C_f \exp\{at + |x|^\beta\}$, with arbitrary fixed numbers $\alpha > 0$, $\beta > 0$.

Let us note that condition (2), for $n > m$, allows “slow” growth of the coefficients $q_k(x)$ as $|x| \rightarrow \infty$. For example, this condition will be fulfilled if

$$|q_k(x)| \leq C(1 + |x|)^{(n-k)m/(n-m)}, \quad 0 \leq k \leq n-1.$$

Obviously, for condition (2) to be fulfilled it is necessary that

$$\int_{-\infty}^{\infty} (1 + |q_k(x)|)^{(m-n)/m(n-k)} dx = \infty, \quad \int_{-\infty}^{\infty} (1 + |q_k(x)|)^{(m-n)/m(n-k)} dx = \infty.$$

Let us also note that for equations of the second order ($n = 2$, $m = 1$) the result of Theorem 1 is valid only under assumptions 1 and 2. Thus the corresponding result from [5] is somewhat strengthened.

We now consider a system of equations of the form (1)

$$\frac{\partial^{m_i} u_i(x, t)}{\partial t^{m_i}} = \sum_{j=1}^N \sum_{k=0}^{n_{ij}} q_{ikj}(x) \frac{\partial^k u_j(x, t)}{\partial x^k}, \quad (7)$$

$$i = 1, \dots, N, \quad t \geq 0; \quad -\infty < x < \infty.$$

Denote

$$\max_i n_{ij} = r_j; \quad \max_j n_{ij} = n_i;$$

$$\max_{ij} n_{ij} = n; \quad \max_i m_i/n_i = 1/p;$$

$$Q_{ij}(x) = \begin{cases} q_{in_{ij}}(x), & \text{if } n_{ij} = n_i, \\ 0, & \text{if } n_{ij} < n_i, \end{cases} \quad R_{ij}(x) = \begin{cases} q_{in_{ij}}(x), & \text{if } n_{ij} = r_j, \\ 0, & \text{if } n_{ij} < r_j. \end{cases}$$

Let $Q_1(x) = \|Q_{ij}(x)\|$, $Q_2(x) = \|R_{ij}(x)\|$.

Theorem 2. Let $H(x) > 0$ be an even monotone function satisfying the condition

$$\int_{-\infty}^{\infty} [H(x)]^{1-p} dx < \infty. \quad (8)$$

Suppose that, for the system (7), the following conditions are fulfilled: 1) $p > 1$; 2) $d_i \geq |\det Q_i(x)| \geq d_0 > 0$, $i = 1, 2$; 3) the coefficients $q_{ikj}(x)$ of the system (7) possess continuous derivatives up to order $n - n_{ij} + k$, and moreover

$$|q_{ikj}^{(r)}(x)| \leq [h(x)]^{n_i - k + r}, \quad -\infty < x < \infty, \quad (9)$$

$i, j = 1, \dots, N$; $k = 0, 1, \dots, n_{ij}$; $r = 0, 1, \dots, n - n_{ij} + k$, where $h(x) > 0$ is an arbitrary even monotone function satisfying the conditions

$$\int_0^\infty [h(x)]^{1-p} dx = \infty; \quad (10)$$

$$xh(x) = o(1) \int_0^x H(t) dt, \quad o(1) \xrightarrow{x \rightarrow \infty} 0. \quad (11)$$

Then uniqueness of the solution of the Cauchy problem for the system (7) in the class of vector functions $\vec{F}(x, t) = \{f_1(x, t), \dots, f_N(x, t)\}$

$$|D_x^j f_k(x, t)| \leq C_F \exp \left\{ \alpha t + \left| \int_0^x H(t) dt \right| \right\},$$

$0 \leq j \leq n - 1$; $k = 1, \dots, N$; $\alpha > 0$ any fixed number, does not hold.

Combining the results of Theorems 1 and 2, we arrive at the following necessary and sufficient condition.

Theorem 3. Let $H(x) > 0$ be an even monotone function; let conditions 1, 2, 3 of Theorem 1 be satisfied and $n > m$.

Then, for uniqueness of the solution of the Cauchy problem (1)–(2) in the class of functions (5), it is necessary and sufficient that the function $H(x)$ satisfy condition (6).

Proofs of Theorems 1 and 2 are based on the following assertions.

Lemma 1. Let the coefficients of the system

$$\sum_{j=1}^N \sum_{k=0}^{n_{ij}} q_{ikj}(x) \frac{d^k V_j(x, \lambda)}{\partial x^k} = \lambda^{m_i} V_i(x, \lambda) \quad (12)$$

satisfy condition 2) of Theorem 2 and condition (9); $\{V_j(x, \lambda)\}$, $j = 1, \dots, N$, is a solution of this system, and

$$|D_x^k V_j(x_0, \lambda)| \leq L, \quad j = 1, \dots, N; \quad k = 0, 1, \dots, r_j - 1.$$

Then

$$|D_x^k V_j(x, \lambda)| \leq LC_1 \exp\{C_2|x|[r^{1/p} + h(x)]\},$$

$$-\infty < x < \infty; \quad r = |\lambda|; \quad C_2 > 0; \quad k = 0, 1, \dots, n - 1; \quad j = 1, \dots, N.$$

Proof of Theorem 2 is based on Lemma 1, from which it follows that every solution $\bar{V}(x, \lambda) = \{V_j(x, \lambda)\}$ of the system (12), satisfying, for example, the initial conditions

$$D_x^k V_j(0, \lambda) = C_{jk} \text{ (const); } \quad k = 0, 1, \dots, r_j - 1; \quad j = 1, \dots, N,$$

is such that

$$\|\bar{V}(x, \lambda)\| = \sup_{\substack{-\infty < x < \infty \\ 1 \leq j \leq N \\ 0 \leq k \leq n-1}} |D_x^k V_j(x, \lambda)| \exp \left\{ - \left| \int_0^x H(t) dt \right| \right\} \leq C_3 \exp \{C_4 \varphi(r)\},$$

where $C_4 > 0$; $H(x)$ is the function indicated in the statement of Theorem 2; $\varphi(r) = r^{1/p} s(r)$; $s(r)$ is determined from the relation

$$H(s(r)) = \delta_1 r^{1/p},$$

$\delta_1 > 0$ is a certain constant. Since

$$\int_0^\infty r^{-2} \varphi(r) dr < \infty,$$

there exists (6) a function $C(\lambda)$, analytic for $\operatorname{Re} \lambda \geq \sigma_0 > 0$, such that

$$\|\bar{V}(x, \lambda) C(\lambda)\| \leq C_5.$$

Hence, by the general theorem of Hill (7), the validity of Theorem 2 follows.

Let $H(x) > 0$ be an even monotone function satisfying condition (6); define the function $g(|\tau|)$ by the relation

$$H(g(|\tau|)) = \delta |\tau|^{m/n}$$

with some constant $\delta > 0$.

Lemma 2. Let the coefficients $q_k(x)$ of the equation

$$aW^n + \sum_{k=0}^{n-2} q_k(x) W^k = (\sigma_1 + i\tau)^m, \quad \sigma_1 > 0, \quad (13)$$

satisfy the condition

$$|q_k(x)| \leq [h(x)]^{n-k}, \quad -\infty < x < \infty, \quad (14)$$

where $h(x) = o(1)H(x)$, $o(1) \rightarrow 0$ as $|x| \rightarrow \infty$. Then equation (13), for $|x| \leq g(|\tau|)$, has n roots of the form

$$W_j(x, \tau) = \varepsilon_j \tau^{m/n} (1 + o(1)),$$

where

$$\varepsilon_j = i^{m/n} a^{-1/n} \exp(2\pi j i / n), \quad j = 0, 1, \dots, n-1; \quad o(1) \rightarrow 0$$

as $|\tau| \rightarrow \infty$.

Denote $B(x, \tau) = \|b_{ij}(x, \tau)\|$, where $b_{ij}(x, \tau) = W_j^{i-1}(x, \tau)$, and write the equation

$$ay^{(n)} + \sum_{k=0}^{n-2} q_k(x) y^{(k)} = \lambda^m y \quad (15)$$

for $\lambda = \sigma_1 + i\tau$, $\sigma_1 > 0$, in the form of a system of first-order equations

$$\bar{y}'(x, \tau) = A(x, \tau) \bar{y}(x, \tau)$$

and set

$$\bar{y}(x, \tau) = B(x, \tau) \bar{z}(x, \tau).$$

Then the vector-function $\bar{z}(x, \tau)$ satisfies the equation

$$\bar{z}'(x, \tau) = W(x, \tau) \bar{z}(x, \tau) + C(x, \tau) \bar{z}(x, \tau),$$

where $W(x, \tau)$ is the diagonal matrix with elements $W_0(x, \tau), \dots, W_{n-1}(x, \tau)$ on the diagonal, and

$$C(x, \tau) = \|c_{ij}(x, \tau)\| = B^{-1}(x, \tau) B'(x, \tau).$$

Lemma 3. If the coefficients $q_k(x)$ of equation (15) satisfy conditions (4) and (14), then, as $|\tau| \rightarrow \infty$,

$$\int_{-g(|\tau|)}^{g(|\tau|)} |c_{ij}(x, \tau)| dx \rightarrow 0.$$

The proof of Lemma 3 is based on an investigation of the structure of the elements $c_{ij}(x, \tau)$ and uses the result of Lemma 2.

Theorem 4. Let the coefficients of equation (15) satisfy conditions (4) and (14) and, in addition, let condition 3 (or condition 4) of Theorem 1 be fulfilled. Then equation (15), for $\lambda = \sigma_1 + i\tau$ and $|x| \leq g(\tau)$, has n linearly independent solutions $y_j(x, \tau)$, $j = 0, 1, \dots, n-1$, such that

$$D_x^k y_j(x, \tau) = (\varepsilon_j \tau^{m/n})^k (1 + o(1)) \exp \int_0^x W_j(t, \tau) dt, \quad (16)$$

$$k = 0, 1, \dots, n-1; \quad o(1) \rightarrow 0$$

as $|\tau| \rightarrow \infty$.

This result is used essentially in the proof of the following theorem.

Theorem 5. Let conditions 1, 2, and 3 of Theorem 1 be fulfilled (or, instead of 3, condition 4), and let the monotone even function $H(x) > 0$ satisfy condition (6). Then every solution $y(x, \lambda)$ of equation (15), analytic for $\operatorname{Re} \lambda \geq \sigma_0 > 0$ and satisfying, for some $C > 0$, the condition

$$\sup_{\substack{-\infty < x < \infty \\ 0 \leq k \leq n-1}} |D_x^k y(x, \lambda)| \exp \left\{ - \left| \int_0^x H(t) dt \right| \right\} < C, \quad (17)$$

is identically equal to zero.

The result of Theorem 1 follows from Theorem 5 and Hill' s theorem ⁽⁷⁾.

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