

CRITERIA FOR DISSIPATIVITY AND THE EXISTENCE OF PERIODIC SOLUTIONS OF PULSED AUTOMATIC SYSTEMS WITH ONE NONLINEAR BLOCK

Yu. A. Dmitriev

1967

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196701.10331>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

UDC 62-503.4

MATHEMATICS

Yu. A. DMITRIEV

CRITERIA FOR DISSIPATIVITY AND THE EXISTENCE OF PERIODIC SOLUTIONS OF PULSED AUTOMATIC SYSTEMS WITH ONE NONLINEAR BLOCK

(Presented by Academician L. S. Pontryagin, 17 X 1966)

For a broad class of pulsed systems with a finite number v of degrees of freedom and one nonlinear block, the forced processes occurring in these systems can be described by difference equations of the form

$$x_{\tau+1} = f(\tau, x_\tau), \quad \tau = 0, \pm 1, \pm 2, \dots,$$

$$f(\tau, x) = Px + q\varphi(\sigma) + k(\tau, x), \quad \sigma = r^*x. \quad (1)$$

Here x, q, r are real $v \times 1$ -vectors (the asterisk denotes transposition); P is a real $v \times v$ -matrix whose spectrum lies inside the unit circle; $\varphi(\sigma)$ is a scalar real continuous function of σ ; $k(\tau, x)$ is a real $v \times 1$ -vector function, continuous in x uniformly with respect to τ , given for all τ , $\tau = 0, \pm 1, \pm 2, \dots$. It is assumed that the nonlinearity $\varphi(\sigma)$ and the function $k(\tau, x)$, which characterizes the external influences, satisfy the conditions

$$\text{a) } \lim_{|\sigma| \rightarrow \infty} \frac{\varphi(\sigma)}{\sigma} \left[a_0 - \frac{\varphi(\sigma)}{\sigma} \right] \geq 0, \quad \text{b) } \beta_1 \leq \frac{\varphi(\sigma_2) - \varphi(\sigma_1)}{\sigma_2 - \sigma_1} \leq \beta_2 \quad (2)$$

for arbitrary values σ_1, σ_2 , where $\beta_1 \leq 0$, $\beta_2 \geq a_0 > 0$ are some prescribed numbers, $\beta_1 \neq -\infty$, $\beta_2 \neq +\infty$;

$$\lim_{|x| \rightarrow \infty} |x|^{-1} k(\tau, x) = 0 \quad \text{uniformly in } \tau. \quad (3)$$

From the frequency characteristic $\chi(e^{i\omega}) = r^*(P - e^{i\omega}I)^{-1}q$ of the linear part of system (1) and the numbers a_0, β_1, β_2 introduced in (2), define the functions

$$\Phi_0(\lambda) = a_0^{-1} + \operatorname{Re} \chi(\lambda);$$

$$\Pi_{\vartheta}(\lambda) = (-1)^{\vartheta} \left[\operatorname{Re}(1 - \lambda)\chi(\lambda) + \frac{1}{2}\beta_{\vartheta}|(1 - \lambda)\chi(\lambda)|^2 \right], \quad \vartheta = 1, 2;$$

$$\theta(\lambda) = |1 - \lambda|^2 [1 + (\beta_1 + \beta_2) \operatorname{Re} \chi(\lambda) + \beta_1 \beta_2 |\chi(\lambda)|^2];$$

$$\Phi_{\vartheta}(\lambda, \eta, \xi) = \Phi_0(\lambda) - \eta \Pi_{\vartheta}(\lambda) + \xi \theta(\lambda), \quad (4)$$

where η, ξ are some real numbers.

Theorem 1. Let $\det \|q : Pq : \dots : P^{\nu-1}q\| \neq 0$. Define the number ϑ ($\vartheta = 1$ or 2) from the condition

$$(-1)^{\vartheta} \lim_{|\sigma| \rightarrow \infty} \sigma^{-2} \left[\int_0^{\sigma} \varphi(\sigma') d\sigma' - \frac{1}{2} \sigma \varphi(\sigma) \right] \geq 0. \quad (5)$$

Suppose that there exist numbers $\eta \geq 0$ and $\xi \geq 0$ such that $\Phi_{\vartheta}(e^{i\omega}, \eta, \xi) > 0$ for all $\omega \in [0, \pi]$. Then system (1) has at least one solution bounded for all τ , $-\infty < \tau < +\infty$, and is dissipative (i.e., in the space $\{x\}$ there exists a bounded closed domain F such that: 1) every solution x_{τ} of system (1), starting from

of some number $\tau = \gamma$, enters this domain, and 2) from $x_{\gamma} \in F$ it follows that $x_{\tau} \in F$ for all $\tau > \gamma$).

Theorem 2. For $\xi = 0$, the assertion of Theorem 1 is valid for any $\vartheta = 1, 2$ without assumption (5), and also in the case when $\det \|q_i P q_j\| \dots \|P^{\nu-1}q\| = 0$.

Theorem 3. Suppose that in equations (1) the nonlinearity $\varphi(\sigma)$ is everywhere differentiable, and $k(\tau, x)$ is a $\nu \times 1$ vector-function periodic in τ with integer period κ . Assume that there exists a number $\eta \geq 0$ for which $\Phi_{\vartheta}(e^{i\omega}, \eta, 0) > 0$ for all ω , $0 \leq \omega \leq \pi$ (see (4)), where either a) $a_0 = \beta_2$, $\vartheta = 1^*$, or b) in the case $\beta_1 = 0$ (see (2)), $\vartheta = 2$ (no additional conditions are imposed on a_0). Then system (1) has at least one solution x_{τ} , κ -periodic for all τ , $-\infty < \tau < +\infty$.

Analogously ^(1,2) it is easy to show that Theorems 1-3 are also valid in the critical case (when some of the eigenvalues of the matrix P are located on the unit circle), if the inequalities $\Phi_{\vartheta} > 0$ are replaced by $\Phi_{\vartheta} \geq 0$ and one requires: a)

$$\lim_{|\sigma| \rightarrow \infty} [\sigma^{-1} \varphi(\sigma) - \varepsilon] [a_0 - \varepsilon - \sigma^{-1} \varphi(\sigma)] \geq 0,$$

where $\varepsilon > 0$ is some fixed (arbitrarily small) number; b) the spectrum of the matrix $P + \alpha q r^*$ is wholly located inside the unit disk for some α , $\alpha \in (0, a_0]$. (The conditions on the eigenvalues of the matrix P are then satisfied automatically.)

If, instead of relations (2a), (3), one has $\varphi(0) = 0$, $0 \leq \sigma^{-1}\varphi(\sigma) \leq a_0$ (for all $\sigma \neq 0$), $k(\tau, x) \equiv 0$, then, according to the results of the author's paper ⁽²⁾, under the conditions of Theorems 1 and 2 the trivial solution $x_\tau \equiv 0$ of system (1) is asymptotically stable in the large in the sense of Lyapunov**.

Finally, we note that Theorems 1-3 are entirely analogous to the corresponding results obtained in the works of V. A. Yakubovich ^(3,4) for the case of continuous-control systems, and supplement the well-known theorem of Ya. Z. Tsytkin ⁽⁵⁾.

It can be shown that the condition $\Phi_\vartheta(e^{i\omega}, \eta, \xi) > 0$ of Theorem 1 covers no less broad a domain in the space of the parameters of system (1) and the numbers a_0, β_1, β_2 than the corresponding criteria ^(1,6), and a broader one than the condition $\Phi_\vartheta(e^{i\omega}, \eta, 0) > 0$, and still more than $\Phi_\vartheta(e^{i\omega}, 0, 0) > 0$.

The proof of Theorems 1-3 is carried out by the method of matrix inequalities of V. A. Yakubovich (see, for example, ^(3,4,8)), extended to the case of discrete systems, and is based on Lemmas 1, 2 ⁽¹⁾, 2, 3 ⁽⁶⁾, Theorem 3 ⁽²⁾, and the following proposition, which is an analogue of the Nyquist criterion***.

Lemma. For a real $\nu \times \nu$ matrix P , real $\nu \times 1$ vectors q, r , a real variable α , $\alpha \in [\alpha_1, \alpha_2]$, and a complex variable λ , define the $\nu \times \nu$ matrix-function $P(\alpha) = P + \alpha q r^*$ and the scalar function $\chi(\lambda) = r^*(P - \lambda I)^{-1}q$. Assume that all eigenvalues of the matrix $P(\alpha_*)$ are located inside the unit disk for some number α_* satisfying the conditions: a) $\alpha_* = 0$, if $0 \in [\alpha_1, \alpha_2]$; b) α_* is an arbitrary number from the interval $[\alpha_1, \alpha_2]$, if $0 \notin [\alpha_1, \alpha_2]$. Then, in order that the spectra of all matrices $P(\alpha)$, $\alpha_1 \leq \alpha \leq \alpha_2$, be located inside the unit disk****, it is necessary and sufficient that the condition

$$1 + \alpha \chi(e^{i\omega}) \neq 0$$

hold for all $\alpha \in [\alpha_1, \alpha_2]$ and $\omega \in [0, \pi]$.

* In the sense of conditions (2), the quantity a_0 in relation (2a), obviously, may be increased (in particular, to the value $a_0 = \beta_2$). However, in this case the condition $\Phi_\vartheta > 0$, by virtue of (4), becomes more stringent.

** In the paper ⁽²⁾, in the author's opinion, a condition of the form (5) was omitted and there is an inaccuracy in the corresponding place in the proof of Theorem 1.

*** This proposition was used in papers ^(1,2,6), but was not clearly formulated there. An analogous proposition in a somewhat different (geometric) interpretation was proved in ⁽¹¹⁾.

**** This condition on the eigenvalues of the matrices $P(\alpha)$ is equivalent to saying that all linear systems $x_{\tau+1} = P(\alpha)x_\tau$, $\alpha \in [\alpha_1, \alpha_2]$, are asymptotically stable in the sense of Lyapunov.

It can be shown that in the case $a_1 = 0$, $a_2 = a_0$ the last relation always holds if the condition $\Phi_\vartheta > 0$ ($\xi \geq 0$, $\eta \geq 0$) of Theorems 1-3 is satisfied.

The **proof of the lemma** can be carried out as, in the analogous case of continuous control systems, this was done in ⁽⁷⁾, or by means of certain techniques of ⁽⁹⁾.

The **proof of Theorem 1** uses the following technique of ^(2,3). Introduce the notation

$$z = \left\| \frac{x}{\varphi(\sigma)} \right\|, \quad Q = \left\| \frac{P}{0} \mid \frac{q}{1} \right\|, \quad d = \left\| \frac{0}{1} \right\|, \quad r_1 = \left| \frac{r}{0} \right|, \quad n(\tau, z) = \left\| \frac{k(\tau, x)}{0} \right\|,$$

$$r_2 = (Q^* - I)r_1, \quad \sigma = r^*x = r_1^*z, \quad \Delta\sigma = r^*f(\tau, x) - r^*x, \quad (6)$$

$$\zeta(\tau, z) = \Delta\varphi = \varphi(\sigma + \Delta\sigma) - \varphi(\sigma), \quad f_1(\tau, z) = Qz + d\zeta(\tau, z) + n(\tau, z),$$

where the quantities $f, x, \varphi, P, q, r, k$ are the same as in equations (1). It is easy to verify that equations (1) are equivalent to the system

$$z_{\tau+1} = f_1(\tau, z_\tau), \quad \tau = 0, \pm 1, \pm 2, \dots$$

Consider the Lyapunov function

$$V(x) \equiv V_1(z) = z^*Hz + \eta(-1)^\vartheta \int_0^{r_1^*z} \varphi(\sigma) d\sigma, \quad H = H^*, \quad (7)$$

where the $(\nu+1) \times 1$ -vectors z, r_1 have the form (6), and the real $(\nu+1) \times (\nu+1)$ -matrix H and the parameters η, ϑ ($\vartheta = 1$ or 2) are to be chosen in accordance with Lemma 1 ⁽⁴⁾.

According to the above, the discrete “derivative” $\Delta V \equiv V[f(\tau, x)] - V(x)$ of the function (7), taken by virtue of equations (1), can be computed from the formula $\Delta V = V_1[f_1(\tau, z)] - V_1(z)$. Add to the obtained result the identically zero quantity*

$$\Lambda = -z^*Rz + 2z^*s\zeta - \xi\zeta^2 + \left[\frac{1}{2}\eta(-1)^\vartheta\beta_\vartheta - \xi\beta_1\beta_2 \right] n^*r_1r_1^*n + z^*[\eta(-1)^\vartheta(dr_1^* + \beta_\vartheta r_2r_1^*) - 2\xi\beta_1\beta_2r_2r_1^*]n$$

$$+\xi(\beta_1+\beta_2)r_1^*n\zeta-\varphi(\sigma-a_0^{-1}\varphi)-\eta(-1)^\vartheta\left(\varphi\Delta\sigma+\frac{1}{2}\beta_\vartheta\Delta\sigma^2\right)-\xi(\Delta\varphi-\beta_1\Delta\sigma)(\beta_2\Delta\sigma-\Delta\varphi),$$

where

$$R=R^*=a_0^{-1}dd^*-\frac{1}{2}(dr_1^*+r_1d^*)-\frac{1}{2}\eta(-1)^\vartheta(dr_2^*+r_2d^*)+\left[\xi\beta_1\beta_2-\frac{1}{2}\eta(-1)^\vartheta\beta_\vartheta\right]r_2r_2^*,$$

$$s=\frac{1}{2}\xi(\beta_1+\beta_2)r_2,$$

the numbers a_0, β_1, β_2 occur in (2), (4), and the real parameter ξ is to be chosen. With the aid of (6) we find

$$-\Delta V=\Psi(z,\zeta)+\Omega(\tau,x).$$

Here

$$\Psi(z,\zeta)=z^*(G+R)z+2z^*g\zeta+\gamma\zeta^2,\quad G=H-Q^*HQ,$$

$$-g=Q^*Hd+s,\quad \gamma=\xi-d^*Hd,$$

and the function $\Omega(\tau,x)$ is easily estimated by means of relations (2), (3) and the inequalities $\eta\geq 0, \xi\geq 0$. Namely: for any $\varepsilon>0$ there exist some positive, finite numbers $\omega_0, \omega_1, \xi_0$ such that

$$\Omega\geq-\varepsilon|x|^2-\omega_1|x|-\omega_0$$

for all $x, |x|\geq \xi_0$.

By Theorem 3 ⁽²⁾, the conditions of Theorem 1 guarantee the existence of a matrix $H=H^*$ for which $\Psi(z,\zeta)>0$ for all $z,\zeta, |z|+|\zeta|\neq 0$.

Substituting the parameters η, ϑ appearing in the conditions of Theorem 1 and the matrix H found above into expression (7), we obtain from the inequality $\Psi>0$ the existence of numbers $\varepsilon>0, \varepsilon_1>0$ such that

$$\Psi(z,\zeta)\geq 2\varepsilon|z|^2+\varepsilon_1\zeta^2,$$

and, all the more (see (6)),

$$\Psi(z,\zeta)\geq 2\varepsilon|x|^2.$$

Taking into account the estimate of the function Ω , we obtain

$$-\Delta V \geq \varepsilon|x|^2 - \omega_1|x| - \omega_0$$

for all x , $|x| \geq \xi_0$, i.e. the function (7) satisfies conditions a), b) of Lemma 1⁽¹⁾. Representing, by means of (6), the function (7) in the form

$$V = x^* H_0 x + 2x^* h_0 \varphi + \chi_0 \varphi^2 +$$

* This expression was obtained in accordance with the S -procedure (see (3,7,8)), which reduces the problem o

$$+ \eta(-1)^\vartheta \int_0^{r^* x} \varphi(\sigma) d\sigma,$$

where $H_0 = H_0^*$ is a certain $\nu \times \nu$ matrix, h_0 is a $\nu \times 1$ vector, and χ_0 is a number; with the aid of the lemma formulated in the present paper and Lemma 3⁽⁶⁾, one can show, analogously to how this was done in⁽⁶⁾, that, by virtue of condition (5) of Theorem 1, the relation $\lim_{|x| \rightarrow \infty} V(x) = +\infty$ holds. Thus the function (7) satisfies condition c) of Lemma 1⁽¹⁾. From what has been proved, by Lemma 1⁽¹⁾, we obtain the assertions of Theorem 1.

The **proof of Theorems 2 and 3** is carried out with the aid of the Lyapunov function

$$V(x) = x^* H x + \eta(-1)^\vartheta \int_0^{r^* x} \varphi(\sigma) d\sigma, \quad H = H^*, \quad (8)$$

where the real $\nu \times \nu$ matrix H and the parameters η, ϑ ($\vartheta = 1$ or 2) are to be chosen in accordance with Lemma 1⁽¹⁾.* Analogously to the preceding discussion, from (1), (8) we find the discrete "derivative" $\Delta V \equiv V[f(\tau, x)] - V(x)$ and add to the resulting expression the quantity

$$-x^* R x + 2x^* s \varphi - \rho \varphi^2 - \varphi(\sigma - \alpha_0^{-1} \varphi) - \eta(-1)^\vartheta (\varphi \Delta \sigma + \frac{1}{2} \beta_\vartheta \Delta \sigma^2) \equiv 0,$$

where

$$R = R^* = -\frac{1}{2} \eta(-1)^\vartheta \beta_\vartheta (I - P^*) r r^* (I - P),$$

$$s = \frac{1}{2} [I - \eta(-1)^\vartheta (1 + \beta_\vartheta r^* q) (I - P^*)] r,$$

$$\rho = \alpha_0^{-1} - \frac{1}{2} \eta(-1)^\vartheta r^* q (2 + \beta_\vartheta r^* q),$$

the quantity $\Delta \sigma$ is defined in (6), and the numbers $\alpha_0, \beta_1, \beta_2$ occur in relations (2), (4). Then ΔV can be represented in the form

$$-\Delta V = \Psi(x, \varphi) + \Omega(\tau, x),$$

where

$$\begin{aligned}\Psi(x, \varphi) &= x^*(G + R)x + 2x^*g\varphi + \gamma\varphi^2, \\ G &= H - P^*HP, \quad -g = P^*Hq + s, \quad \gamma = \rho - q^*Hq,\end{aligned}$$

and for the function Ω the same estimate is valid as in the case of Theorem 1. With the aid of Lemma 2 ⁽⁶⁾ one can show that the conditions of Theorems 2 and 3 guarantee the existence of a matrix $H = H^*$ for which $\Psi(x, \varphi) > 0$ for all x, φ , $|x| + |\varphi| \neq 0$. Therefore, analogously to the case of Theorem 1, the function (8) (with the indicated matrix H) satisfies requirements a), b) of Lemma 1 ⁽¹⁾. Now, using the lemma formulated in the present paper, Lemma 3 ⁽⁶⁾, and certain techniques from ⁽⁴⁾, it is easy to verify that under the conditions of Theorem 2 the function (8) satisfies the requirement $\lim_{|x| \rightarrow \infty} V(x) = +\infty$, and under the conditions of Theorem 3, in addition, the inequality $\partial^2 V / \partial x^2 > 0$ for all x . Here $\partial^2 V / \partial x^2$ is the $\nu \times \nu$ matrix of second derivatives of V with respect to the components of the vector x . From what has been proved, by Lemmas 1, 2 ⁽¹⁾, as well as in ⁽¹⁾, the assertions of Theorems 2, 3 follow.

Siberian Scientific-Research
Institute of Power Engineering

Received
26 VII 1966

REFERENCES

1. Yu. A. Dmitriev, DAN, 164, No. 1 (1965).
2. Yu. A. Dmitriev, DAN, 164, No. 2 (1965).
3. V. A. Yakubovich, DAN, 160, No. 2 (1965).
4. V. A. Yakubovich, Avtomatika i telemekhanika, 25, No. 7 (1964).
5. Ya. Z. Tsypkin, DAN, 152, No. 2 (1964).
6. Yu. A. Dmitriev, DAN, 160, No. 3 (1965).
7. M. A. Aizerman, F. R. Gantmakher, Absolute Stability of Nonlinear Control Systems, Moscow, Academy of Sciences of the USSR, 1963.
8. F. R. Gantmakher, V. A. Yakubovich, Proceedings of the Second All-Union Congress on Theoretical and Applied Mechanics, Moscow, January 1964, Nauka Publishing House, 1965.
9. V. M. Popov, Acad. Rep. Pop. Romîne, Inst. de Energetica, Studii si cercetări de energetica, 9, No. 3 (1959).

10. G. P. Szegő, C. R., 257, No. 11 (1963).
11. Ya. Z. Tsypkin, Avtomatika i telemekhanika, 25, issue 3 (1964).

* A frequency condition for the absolute stability of equations of the form (1) with the aid of a function analogous to (8) was first obtained by Szegő ⁽¹⁰⁾ for the case when, instead of conditions (2a), (3), the following are satisfied:

$$\varphi(0) = 0, \quad 0 < \sigma^{-1}\varphi(\sigma) \leq \alpha_0 \quad \text{for } \sigma \neq 0, \quad k(\tau, x) \equiv 0.$$

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.