

IMPROVING THE SMOOTHNESS OF FUNCTIONS IN A. N. KOLMOGOROV' S THEOREM ON SUPERPOSITIONS

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Abstract

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MATHEMATICS

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IMPROVING THE SMOOTHNESS OF FUNCTIONS IN A. N. KOLMOGOROV'S THEOREM ON SUPERPOSITIONS

(Presented by Academician A. N. Kolmogorov on 20 II 1967)

By A. N. Kolmogorov's theorem ⁽¹⁾, every continuous function of n variables can be represented in the form

$$f(x_1, x_2, \dots, x_n) = \sum_{i=1}^{2n+1} \chi_i \left(\sum_{j=1}^n \varphi_{i,j}(x_j) \right), \quad (\text{A})$$

where all the functions are continuous, and the $\varphi_{i,j}(x_j)$ are fixed and do not depend on the function being decomposed $f(x_1, x_2, \dots, x_n)$. G. M. Henkin showed that the $\varphi_{i,j}(x_j)$ can be chosen to satisfy a Hölder condition with any exponent $0 < \alpha < 1$. A. G. Vitushkin and G. M. Henkin proved that if continuously differentiable functions are fixed as the $\varphi_{i,j}(x_j)$, then the set of superpositions of the form (A) is nowhere dense in the space of all continuous functions ^(2,3). Here we shall show that the inner functions in A. N. Kolmogorov's construction can be chosen to satisfy a Lipschitz condition. The precise formulation of the result is as follows.

Theorem. Every function continuous on the cube $0 \leq x_j \leq 1$ ($j = 1, 2, \dots, n$) can be represented in the form

$$f(x_1, x_2, \dots, x_n) = \sum_{i=1}^{2n+1} \chi_i \left(\sum_{j=1}^n \varphi_{i,j}(x_j) \right),$$

where $\chi_i(x)$ are continuous functions, and $\varphi_{i,j}(x_j)$ are preassigned fixed functions, independent of the function $f(x_1, x_2, \dots, x_n)$, and satisfying a Lipschitz condition.

For simplicity of exposition we shall give the proof for $n = 2$. The proof will use the following lemma, whose proof may be found in ⁽¹⁾.

Lemma 1. (*A. N. Kolmogorov*). Let $P_{\alpha,j}^k$ ($k = 1, 2, \dots, 5$; $\alpha = 1, 2, \dots, \alpha_{j,k}$; $j = 1, 2, \dots$) be closed rectangles in the square I_2 ($0 \leq x \leq 1$, $0 \leq y \leq 1$), satisfying the following conditions:

1. For all k and j , if $\alpha_1 \neq \alpha_2$, then

$$P_{\alpha_1,j}^k \cap P_{\alpha_2,j}^k = \emptyset.$$

2. The diameter of the rectangle $P_{\alpha,j}^k$ tends to zero as the index j increases.
3. For every fixed j , every point of the square I_2 belongs to at least three rectangles from the system $\{P_{\alpha,j}^k\}$ with one and the same fixed index j .

Suppose, further, that continuous functions $\Phi_k(x, y)$ are given such that, for all k and j , the sets $\{\Phi_k(P_{\alpha,j}^k)\}$ ($\alpha = 1, 2, \dots, \alpha_{j,k}$) are pairwise disjoint.

Then, for every continuous function $f(x, y)$ on I_2 , one can specify continuous functions $\chi_k(x)$ such that

$$f(x, y) = \sum_{k=1}^5 \chi_k(\Phi_k(x, y)).$$

For the proof of the theorem it suffices to construct a system of rectangles $P_{\alpha,j}^k$ and functions $\Phi_k(x, y) = \varphi_k(x) + \psi_k(y)$ satisfying the Lipschitz conditions and Lemma 1.

Construction of the systems of rectangles. On the segment I_1 ($0 \leq x \leq 1$) we construct a system $\{S_{i,j}^k\}$ of intervals $S_{i,j}^k$ ($k = 1, 2, \dots, 5$; $i = 1, 2, \dots, i_{j,k}$; $j = 1, 2, \dots$) of length $\Delta_{i,j}^k$, possessing the following properties:

- a) for any k and j , if $i_1 \neq i_2$, then $S_{i_1,j}^k \cap S_{i_2,j}^k = \emptyset$;
- b) $\Delta_{i,j}^k \rightarrow 0$ as $j \rightarrow \infty$;
- c) for every j , each point of I_1 belongs to at least 4 intervals of the system $\{S_{i,j}^k\}$ with one and the same fixed j .

The construction of $S_{i,j}^k$ will be carried out by induction on j . For $j = 1$ we take $i_{1,k} = 1$ and put $S_{1,1}^k = I_1$. We shall assume that the intervals $S_{i,n-1}^k$ have been constructed. From the set $\{S_{i,n-1}^k\}$ choose an interval $S_{i_1,n-1}^{k_1}$ such that

$$\Delta_{i_1,n-1}^{k_1} \geq \Delta_{i,n-1}^k$$

for all i and k . Near its center, for example at a distance not exceeding $\frac{1}{4}\Delta_{i_1,n-1}^{k_1}$, fix a point $x_{i_1,n-1}^{k_1}$ which is not an endpoint of the intervals $\{S_{i,n-1}^k\}$. By property c) this point belongs to either 4 or 5 intervals of the system $\{S_{i,n-1}^k\}$. Having fixed $x_{i_1,n-1}^{k_1}$, we arrive at one of the following two cases.

1. $x_{i_1, n-1}^{k_1}$ belongs to 5 intervals $\{S_{i, n-1}^k\}$. Then a sufficiently small interval $\lambda \subset S_{i_1, n-1}^{k_1}$ with $\lambda \ni x_{i_1, n-1}^{k_1}$ will be covered by these same intervals. Remove from $S_{i_1, n-1}^{k_1}$ a sufficiently small interval belonging to λ , containing the point $x_{i_1, n-1}^{k_1}$, and lying at a positive distance from the endpoints of the intervals of the system $\{S_{i, n-1}^k\}$. We obtain two intervals L and R (left and right). In this case, for $k \neq k_1$ put $S_{i, n}^k = S_{i, n-1}^k$. As $\{S_{i, n}^{k_1}\}$ take the system consisting of the intervals L and R and of the intervals $\{S_{i, n-1}^{k_1}\}$, with the exception of the interval $S_{i_1, n-1}^{k_1}$.
2. $x_{i_1, n-1}^{k_1}$ belongs to 4 intervals $\{S_{i, n-1}^k\}$. This means that for some p ,

$$x_{i_1, n-1}^{k_1} \notin \bigcup_i S_{i, n-1}^p.$$

In this case remove from the segment $S_{i_1, n-1}^{k_1}$ an interval $\lambda \ni x_{i_1, n-1}^{k_1}$ of such small length that it is possible to indicate an interval $B_{i_1, n-1}^p$ such that

$$B_{i_1, n-1}^p \cap \bigcup_i S_{i, n-1}^p = \emptyset$$

and $B_{i_1, n-1}^p \supset \lambda$. Let us note here that the length Δ_n of the interval $B_{i_1, n-1}^p$ can be chosen arbitrarily small. Δ_n will be chosen below in the construction of the functions.

In this case, as before, the intervals L and R (obtained by removing λ from $S_{i_1, n-1}^{k_1}$) and $\{S_{i, n-1}^{k_1}\}$ without the interval $S_{i_1, n-1}^{k_1}$ will constitute $\{S_{i, n}^{k_1}\}$. The system $\{S_{i, n}^p\}$ will consist of the intervals $\{S_{i, n-1}^p\}$ and the interval $B_{i_1, n-1}^p$. For $k \neq k_1$ and $k \neq p$ put $S_{i, n}^k = S_{i, n-1}^k$.

Thus the systems $\{S_{i, j}^k\}$ ($k = 1, 2, \dots, 5$) on I_1 have been constructed. (In the assumption that Δ_n is given each time when this is necessary. It will be seen below that Δ_n is chosen immediately after the $(n-1)$ -st step of constructing the systems of intervals.) They possess properties a), b), and c).

On the segment $\bar{I}_1$ ($0 \leq y \leq 1$), in an analogous way, we construct systems of intervals $\{T_{i, j}^k\}$. (In constructing the systems $\{T_{i, n}^k\}$, in analogy with $L, R, B_{i_1, n-1}^p$, and $x_{i_1, n-1}^{k_1}$, there arise the intervals $\bar{L}, \bar{R}, \bar{B}_{i_1, n-1}^p$ and the point $y_{i_1, n-1}^{k_1}$.)

Denote

$$P_{(i, l), j}^k = \{(x, y) : x \in S_{i, j}^k, y \in T_{l, j}^k\}.$$

Renumber the pairs (i, l) by the index α and put

$$P_{\alpha, j}^k = P_{(i(\alpha), l(\alpha)), j}^k.$$

The system of rectangles $\{P_{\alpha, j}^k\}$ satisfies the conditions of Lemma 1.

Construction of the functions. $\varphi_k(x)$ and $\psi_k(y)$ will be obtained as the limits of certain sequences $\varphi_{k,j}(x)$ and $\psi_{k,j}(y)$ as $j \rightarrow \infty$.

We shall construct $\varphi_{k,j}(x)$ and $\psi_{k,j}(y)$ so that they satisfy the following conditions:

) for fixed j and k , the functions $\varphi_{k,j}(x)$ and $\psi_{k,j}(y)$ are constant on the intervals $S_{i,j}^k$ and $T_{i,j}^k$, respectively, and linear in the intervals complementary to $\bigcup_i S_{i,j}^k$ and $\bigcup_i T_{i,j}^k$, respectively;

) $\varphi_{k,j}(x)$ and $\psi_{k,j}(y)$ are nondecreasing functions;

) the function

$$\Phi_{k,j}(x, y) = \varphi_{k,j}(x) + \psi_{k,j}(y)$$

is such that, for any fixed k and $\tau \leq j$, and for $a_1 \neq a_2$,

$$\rho(\Phi_{k,j}(P_{a_1,\tau}^k), \Phi_{k,j}(P_{a_2,\tau}^k)) > \varepsilon_\tau > 0,$$

where $\Phi_{k,j}(P_{a,\tau}^k)$ is the set of values of the function $\Phi_{k,j}(x, y)$ on the rectangle $P_{a,\tau}^k$, and $\rho(B, C)$ is the distance between the sets B and C ;

) the functions $\varphi_{k,j}(x)$ and $\psi_{k,j}(y)$ satisfy a Lipschitz condition with constant < 1 ;

)

$$\max_x |\varphi_{k,j-1}(x) - \varphi_{k,j}(x)| \leq 1/2^j, \quad \max_y |\psi_{k,j-1}(y) - \psi_{k,j}(y)| \leq 1/2^j$$

(condition () is needed only to ensure the uniform convergence of the functions $\varphi_{k,j}(x)$ and $\psi_{k,j}(y)$ as $j \rightarrow \infty$).

The functions $\varphi_{k,j}(x)$ and $\psi_{k,j}(y)$ are constructed by induction.

For $j = 1$, $k = 1, 2, \dots, 5$, put $\varphi_{k,1}(x) = \psi_{k,1}(y) \equiv 0$. The passage from the functions $\varphi_{k,n-1}(x)$ and $\psi_{k,n-1}(y)$ to the functions $\varphi_{k,n}(x)$ and $\psi_{k,n}(y)$ is connected with the construction of the intervals $\{S_{i,n}^k\}$ and $\{T_{i,n}^k\}$, and is therefore carried out by one of the following two methods.

I. In case 1 (see the construction of the systems of intervals), for $k \neq k_1$ we take

$$\varphi_{k,n}(x) = \varphi_{k,n-1}(x)$$

and

$$\psi_{k,n}(y) = \psi_{k,n-1}(y).$$

For $k = k_1$, on all intervals $S_{i,n}^{k_1}$, except and , put $\varphi_{k_1,n}(x) = \varphi_{k_1,n-1}(x)$, and on $T_{i,n}^{k_1}$, except and , put $\psi_{k_1,n}(y) = \psi_{k_1,n-1}(y)$.

Put

$$\varphi_{k_1,n}() = \varphi_{k_1,n-1}() + \varepsilon_{n,1}, \quad \psi_{k_1,n}() = \psi_{k_1,n-1}() - \varepsilon_{n,2},$$

$$\varphi_{k_1,n}() = \varphi_{k_1,n-1}() + \varepsilon_{n,3}, \quad \psi_{k_1,n}() = \psi_{k_1,n-1}() + \varepsilon_{n,4},$$

and in the intervals complementary to $\bigcup_i S_{i,n}^{k_1}$ and $\bigcup_i T_{i,n}^{k_1}$, respectively, define $\varphi_{k_1,n}(x)$ and $\psi_{k_1,n}(y)$ to be linear and to agree at the endpoints of the intervals with the already assigned values.

Choose the numbers $\varepsilon_{n,\beta} > 0$ so that $\varphi_{k_1,n}(x)$ and $\psi_{k_1,n}(y)$ still satisfy conditions (), (), (), and (). (We note that under condition (), for sufficiently small $\varepsilon_{n,\beta}$, it is necessary to monitor only the case $\tau = n$, since for small changes of $\varphi_{k_1,n-1}(x)$ and $\psi_{k_1,n-1}(y)$ the obtained functions $\varphi_{k_1,n}(x)$ and $\psi_{k_1,n}(y)$ satisfy condition () for $\tau < n$.)

II. In case 2 (see the construction of the systems of intervals), for $k \neq k_1$ and $k \neq p$ put

$$\varphi_{k,n}(x) = \varphi_{k,n-1}(x)$$

and

$$\psi_{k,n}(y) = \psi_{k,n-1}(y).$$

On the intervals

$$S_{i,n}^p \neq B_{i_1,n-1}^p$$

and

$$T_{i,n}^p \neq \overline{B}_{i_1,n-1}^p$$

respectively, take

$$\varphi_{p,n}(x) = \varphi_{p,n-1}(x)$$

and

$$\psi_{p,n}(y) = \psi_{p,n-1}(y).$$

Put

$$\varphi_{p,n}(B_{i_1,n-1}^p) = \varphi_{p,n-1}(x_{i_1,n-1}^{k_1}) + \varepsilon'_n, \quad \psi_{p,n}(\overline{B}_{i_1,n-1}^p) = \psi_{p,n-1}(y_{i_1,n-1}^{k_1}) + \varepsilon''_n,$$

where ε'_n and ε''_n will be chosen below.

In the intervals complementary to $\bigcup_i S_{i,n}^p$ and $\bigcup_i T_{i,n}^p$, respectively, define $\varphi_{p,n}(x)$ and $\psi_{p,n}(y)$ to be linear and to agree at the endpoints of the intervals with the already assigned values. Recall that we could choose the length Δ_n of the intervals $B_{i_1,n-1}^p$ and $\overline{B}_{i_1,n-1}^p$ arbitrarily. We now choose it so that, for sufficiently small ε'_n and ε''_n , $\varphi_{p,n}(x)$ and $\psi_{p,n}(y)$ satisfy condition (). Now choose ε'_n and ε''_n so that the functions $\varphi_{p,n}(x)$ and $\psi_{p,n}(y)$ satisfy conditions (), (), (), and ().

The functions $\varphi_{k_1,n}(x)$ and $\psi_{k_1,n}(y)$ are obtained from the functions $\varphi_{k_1,n-1}(x)$ and $\psi_{k_1,n-1}(y)$ in the same way as in case 1.

Now set

$$\varphi_k(x) = \lim_{j \rightarrow \infty} \varphi_{k,j}(x), \quad \psi_k(y) = \lim_{j \rightarrow \infty} \psi_{k,j}(y),$$

$$\Phi_k(x, y) = \varphi_k(x) + \psi_k(y).$$

Since for $\Phi_{k,j}(x, y)$ condition (e) is fulfilled for every j , the $\Phi_k(x, y)$ satisfy the conditions of Lemma 1. From condition () it follows that $\varphi_k(x)$ and $\psi_k(y)$, for every k , satisfy a Lipschitz condition with constant 1. The theorem is proved.

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Note: Figure translations are in progress. See original paper for figures.

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