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Abstract

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MATHEMATICS

B. EFIMOV

ON EXTREMALLY DISCONNECTED BICOMPACTA

(Presented by Academician P. S. Aleksandrov, 5 IV 1966)

P. S. Aleksandrov, at a seminar at Moscow University, posed the following problem: does there exist a universal extremally disconnected bicom pactum of weight τ^* , topologically containing every other extremally disconnected bicom pactum of weight $\leq \tau$? In this note it is proved (Theorem 1) that for all admissible cardinal numbers such a bicom pactum exists. Here the cardinal number τ is called admissible if $\tau^{\aleph_0} = \tau$. It is further proved (Theorem 2) that this bicom pactum is homogeneous with respect to character^{**}. This answers a question proposed by M. Ya. Antonovskii: do there exist extremally disconnected bicom pacta homogeneous with respect to character? Finally, Theorem 3 asserts that the weight of any infinite extremally disconnected bicom pactum is an admissible cardinal number, which completely solves both problems.

§ 1. V. I. Ponomarev ⁽¹⁻³⁾ and A. Gleason ⁽⁴⁾ proved that for every completely regular (Gleason—for locally bicom pact) space X there exists an irreducible perfect^{***} extremally disconnected preimage pX , which the former called an absolute and the latter a projective space. For a bicom pact space X , the simplest construction is that of S. Iliadis ^(5,6), S. V. Fomin ⁽⁷⁾, which consists in the following. The points of pX are declared to be all possible maximal centered systems $\mathfrak{F}$ consisting of open subsets of X . As a base of closed sets in pX one takes sets of the form $\Gamma_U = \mathcal{E}(\mathfrak{F} : U \in \mathfrak{F})$, if U is open in X . It can be proved that the space pX is bicom pact and extremally disconnected. Assigning to each point $x \in pX$ (= a maximal centered system $\mathfrak{F}$) its unique point of contact from X (X a bicom pactum), we obtain an irreducible mapping of pX onto X .

Lemma 1. *The weight of the absolute pX of a bicom pactum X is equal to the cardinality of all canonically closed subsets of X^{**} .*

Proof. Let $f : pX \rightarrow X$ be an irreducible mapping of pX onto X . To each canonically closed $[U] \subset X$ we assign

$$V = [f^{-1}U]_{pX},$$

a certain open-and-closed subset of pX . It can be proved that, by virtue of the irreducibility and perfection of the mapping f , this rule establishes a one-to-one

correspondence between all open-and-closed subsets of pX and all canonically closed subsets of X . Since the cardinality of any open-and-closed base of pX is equal to the weight of pX , the lemma follows.

* A topological space X is called **extremally disconnected** if the closure $[U]_X$ of any open $U \subset X$ is again open. The minimum of the cardinalities of open bases of the space X is called the **weight** and is denoted by wX .

** The **character** $\chi(F, X)$ of a set F in X is the minimum of the cardinalities of fundamental systems of neighborhoods of F in X .

*** A continuous mapping $f : X \rightarrow Y$ is **irreducible** if there is no proper closed subset $F \subset X$ for which $f(F) = Y$. A mapping is called **perfect** if it is closed and bicompat.

**** A set $F = [U]$ is called **canonically closed** if it is the closure of a nonempty open U .

Definition. A cardinal number τ will be called **admissible** if $\tau^{\aleph_0} = \tau$.

Theorem 1. Let τ be an admissible cardinal number, $D^\tau = \prod D_\alpha^{0,1}$, $\alpha \in A$, $|A| = \tau^*$, be the generalized Cantor discontinuum of weight τ . Then the absolute pD^τ is a universal extremally disconnected bicompatum of weight τ .

Proof. First of all, let us note that pD^τ is an extremally disconnected bicompatum of weight τ . Indeed, in the space D^τ every canonical closed set has type G_δ (Theorem 5⁽⁸⁾). By virtue of bicompatness and zero-dimensionality of D^τ , every closed set of type G_δ in D^τ is the intersection of a countable number of open-and-closed sets. Since the cardinality of all open-and-closed sets in D^τ is τ , the cardinality of all closed sets of type G_δ in D^τ is $\tau^{\aleph_0} = \tau$ (τ is admissible!). Applying Lemma 1, we obtain that $w(pD^\tau) = \tau$. Let us prove the universality of pD^τ . Let X be an arbitrary extremally disconnected bicompatum of weight $\leq \tau$. By a theorem of N. B. Vedenisov⁽⁹⁾, the bicompatum X , as a zero-dimensional space in the sense of ind, can be topologically embedded in D^τ . Let $X \subset D^\tau$ and let $f : pD^\tau \rightarrow D^\tau$ be irreducible. Then $Y = f^{-1}X \subset pD^\tau$. By Brawer's theorem^(10, p. 27), in pD^τ there exists $X^* \subset Y \subset pD^\tau$ such that $f(X^*) = X$, and on X^* the mapping f is irreducible. This means, by virtue of the extremal disconnectedness of X , that X^* is homeomorphic to X (Lemma 2.3⁽⁴⁾), as was required to prove.

§ 2. A system $\mathfrak{B} = \{[U]\}$ consisting of nonempty canonical closed sets will be called a **δ -system of the closed set** $F \subset R$, if: 1) for any two $[U], [V] \in \mathfrak{B}$ we have $[U] \cap [V] = [W] \in \mathfrak{B}$, and 2) $\cap[U] = F$ for all $[U] \in \mathfrak{B}$. The minimum of the cardinalities of all δ -systems of the closed set $F \subset R$ will be called the **δ -character** of F in R , and will be denoted by $\delta(F, R)$. If R is a compact metric space, then $\delta(F, R) \leq \psi(F, R) = \chi(F, R)^{**}$. In general, only the inequality holds

$$\delta(F, R) \leq \psi(F, R) \leq \chi(F, R).$$

- a) Let $f : X \rightarrow Y$ be an irreducible mapping of the bicomactum X onto Y . Then $\delta(F, X) \geq \delta(f, F, Y)$.

This proposition follows from the fact that the irreducible image of a δ -system F is a δ -system fF .

- b) For all $x \in D^\tau$, we have $\delta(x, D^\tau) = \psi(x, D^\tau) = \chi(x, D^\tau)$.

Let us prove b). Since $\delta(x, D^\tau) \leq \psi(x, D^\tau) = \chi(x, D^\tau) = \tau$ ⁽⁸⁾, it is enough to prove that $\tau \leq \delta(x, D^\tau)$. Let $\mathfrak{B} = \{[U]\}$ be a minimal δ -system of the point $x \in D^\tau$. Since in D^τ every canonical closed set has type G_δ (Theorem 5 ⁽⁸⁾), for every $[U] \in \mathfrak{B}$ we have $[U] = \bigcap_{k=1}^{\infty} O_k[U]$; hence

$$x = \bigcap [u] = \bigcap \bigcap_{k=1}^{\infty} O_k[U],$$

where the first intersection is taken over all $[U] \in \mathfrak{B}$. Thus we obtain that $\tau = \psi(x, D^\tau) \leq \aleph_0 |\mathfrak{B}| = |\mathfrak{B}|$, as was required to prove.

Theorem 2. Let τ be an admissible cardinal number. Then pD^τ is an extremally disconnected bicomactum homogeneous with respect to character τ^{**} .

Proof. Since τ is an admissible cardinal number, by Theorem 1 we have $\chi(x, pD^\tau) \leq w(pD^\tau) = \tau$. Therefore it is enough to prove that $\chi(x, pD^\tau) \geq \tau$. Suppose that $\chi(x, pD^\tau) = \mathfrak{m} < \tau$. Let $f : pD^\tau \rightarrow D^\tau$ be irreducible; then, by property a), we have $\mathfrak{m} = \chi(x, pD^\tau) \geq \delta(x, pD^\tau) \geq \delta(fx, D^\tau)$. By property b), $\delta(fx, D^\tau) = \chi(fx, D^\tau) = \tau$. Thus—

* $|A|$ is the cardinality of the set A .

** The **pseudocharacter** $\psi(F, X)$ of the set F in X is the least cardinal number $\mathfrak{m}$ such that $F = \bigcap O_\alpha$, $\alpha \in A$, all O_α are open and $|A| = \mathfrak{m}$. If F is closed and X is a bicomactum, then $\chi(F, X) = \psi(F, X)$.

*** A space X is called **homogeneous with respect to character** τ if $\chi(x, X) = \tau$ for all $x \in X$.

Thus we have obtained that $\mathfrak{m} \geq \tau$, contrary to the assumption. The theorem is proved.

§ 3. The sign $\oplus$ denotes the disjoint union of spaces X_k ; βX denotes the Stone-Čech compactification of X . Let

$$X = \bigoplus_{k=1}^{\infty} X_k$$

and $f_k : X_k \rightarrow Y_k$; then by

$$f = \bigoplus_{k=1}^{\infty} f_k$$

we denote the mapping

$$f : X \rightarrow \bigoplus_{k=1}^{\infty} Y_k,$$

for which the restriction $f|_{X_k} = f_k$.

Lemma 2. Let $\{\mathfrak{m}_k\}$ be a countable set of infinite cardinal numbers and

$$\tau = \prod_{k=1}^{\infty} \mathfrak{m}_k.$$

Then $\tau^{\aleph_0} = \tau$.

Proof. It is enough to prove that $\tau^{\aleph_0} \leq \tau$. Since

$$\mathfrak{m}_k \leq \sum_{k=1}^{\infty} \mathfrak{m}_k,$$

we have

$$\tau = \prod_{k=1}^{\infty} \mathfrak{m}_k \leq \prod \left(\sum_{k=1}^{\infty} \mathfrak{m}_k \right) = \left(\sum_{k=1}^{\infty} \mathfrak{m}_k \right)^{\aleph_0}.$$

Therefore, using the distributivity of product with respect to sum ((12), p. 191), we obtain

$$\begin{aligned} \tau^{\aleph_0} &\leq \left(\sum_{k=1}^{\infty} \mathfrak{m}_k \right)^{\aleph_0 \cdot \aleph_0} = \left(\sum_{k=1}^{\infty} \mathfrak{m}_k \right)^{\aleph_0} = \sum_{\alpha \in A} \prod_{i=1}^{\infty} \mathfrak{m}_{k(i)}, \quad \alpha \leq |A| \prod_{k=1}^{\infty} \mathfrak{m}_k = \\ &= \aleph_0^{\aleph_0} \prod_{k=1}^{\infty} \mathfrak{m}_k = \prod_{k=1}^{\infty} (\aleph_0 \cdot \mathfrak{m}_k) = \prod_{k=1}^{\infty} \mathfrak{m}_k = \tau. \end{aligned}$$

The lemma is proved.

Lemma 3. Let

$$X = \bigoplus_{k=1}^{\infty} X_k,$$

where the X_k are bcompacta and $wX_k \geq \aleph_0$. Then

$$w(\beta X) = \prod_{k=1}^{\infty} wX_k.$$

Proof. Put $wX_k = \mathfrak{m}_k$ and

$$\prod_{k=1}^{\infty} \mathfrak{m}_k = \tau.$$

We first prove that $w(\beta X) \leq \tau$. For this it is enough to prove that the cardinality of the set of all bounded continuous functions on X does not exceed τ . Denote this set by $\mathfrak{N} = \{f\}$. Let $f_k = f|_{X_k}$ be the restriction of an arbitrary function $f \in \mathfrak{N}$ to X_k ; then

$$f = \bigoplus_{k=1}^{\infty} f_k.$$

Conversely, a combination of arbitrary continuous functions f_k on X_k gives some continuous function on X . Hence

$$|\mathfrak{N}| \leq \prod_{k=1}^{\infty} |C(X_k)|,$$

if $C(X_k)$ is the ring of all continuous real-valued functions on X_k . Since

$$wC(X_k) = wX_k$$

(11) and $C(X_k)$ is a metric space, it follows that

$$|C(X_k)| \leq (wX_k)^{\aleph_0} = \mathfrak{m}_k^{\aleph_0}.$$

Further, applying Lemma 1, we obtain

$$|\mathfrak{N}| \leq \prod_{k=1}^{\infty} \mathfrak{m}_k^{\aleph_0} = \left(\prod_{k=1}^{\infty} \mathfrak{m}_k \right)^{\aleph_0} = \prod_{k=1}^{\infty} \mathfrak{m}_k = \tau.$$

Thus the inequality $w(\beta X) \leq \tau$ is proved. We now prove that $w(\beta X) \geq \tau$. Since $wC(\beta X) = w(\beta X)$, to prove this inequality it is enough to find in $C(\beta X)$ a family $\mathfrak{M} = \{f\}$, $|\mathfrak{M}| \geq \tau$, consisting of functions whose pairwise distances are ≥ 1 . Since $wC(X_k) = wX_k = \mathfrak{m}_k$, in $C(X_k)$ there exists a family $\mathfrak{M}_k = \{f_k\}$ of functions such that*: 1) $\rho(f_k, g_k) = \sup_{x \in X_k} |f_k(x) - g_k(x)| = 1$, $f_k \neq g_k \in \mathfrak{M}_k$; 2) $\sup_{x \in X_k} |f_k(x)| = 1$, $f_k \in \mathfrak{M}_k$; 3) $|\mathfrak{M}_k| = \mathfrak{m}_k$.

* If X_k is zero-dimensional, then for $\mathfrak{M}_k$ one may take the family of functions f for which $f(V) = 0$ and $f(X_k \setminus V) = 1$, where V is an arbitrary open-and-closed subset of X_k .

Next put $f = \bigoplus_{k=1}^{\infty} f_k$, if $f_k \in \mathfrak{M}_k$. Let f^* be the Stone-Ćech extension of the function f to βX , which exists since f is bounded. Then the family $\mathfrak{M} = \{f^*\}$, which is obtained from $\{f\}$ if the f_k independently range over the $\mathfrak{M}_k$, is the desired one. Indeed,

$$|\mathfrak{M}| = \prod_{k=1}^{\infty} |\mathfrak{M}_k| = \prod_{k=1}^{\infty} \mathfrak{m}_k = \tau;$$

further, for any distinct $f^*, g^* \in \mathfrak{M}$ we have

$$\rho(f^*, g^*) = \rho(f, g) = \sup_k \sup_{x \in X_k} |f_k(x) - g_k(x)| = 1,$$

since for some X_k necessarily $f_k \neq g_k$. The lemma is proved.

Theorem 3. *The weight of an infinite extremally disconnected bicom pactum is an admissible cardinal number.*

Proof. Let R be an extremally disconnected bicom pactum, $|R| \geq \aleph_0$, and $wR = \tau$. Construct open-and-closed sets $U_k \subset R$ such that

$$U_k \cap U_n = \emptyset, \quad \text{if } k \neq n, \quad w(U_k) \geq \aleph_0, \quad \text{and} \quad \left[\bigcup_{k=1}^{\infty} U_k \right] = R.$$

Let T be the set of all isolated points in R^* . Consider two cases: 1) $[T] = R$ and 2) $[T] \neq R$. In the first case $|T| \geq \aleph_0$, hence $T = \bigoplus_{n=1}^{\infty} A_n$, $A_n \subset T$ and $|A_n| \geq \aleph_0$. Then $[A_n]_R = U_n$ are the required sets. In the second case the sets $[T]$ and $R \setminus [T]$ are open-and-closed, and $F = R \setminus [T]$ is extremally disconnected and contains no isolated points. By transfinite induction one can construct open-and-closed V_α , $\alpha \in A$, $|A| \geq \aleph_0$, such that $V_\alpha \subset F$, $V_\alpha \cap V_\beta = \emptyset$, if $\alpha \neq \beta$, $wV_\alpha > \aleph_0$, and

$$\left[\bigcup_{\alpha \in A} V_\alpha \right] = F.$$

Represent $A = \bigoplus_{n=1}^{\infty} A_n$, if $A_n \neq \emptyset$, $A_n \cap A_k = \emptyset$. In this case

$$U_n = \left[\bigcup_{\alpha \in A_n} V_\alpha \right]$$

are the required sets^{**}. Thus, in both cases we have constructed an extremally disconnected space

$$X = \bigoplus_{n=1}^{\infty} U_n, \quad wU_n \geq \aleph_0,$$

and the U_n are bicomacta, which is dense in R . Consequently, by Gleason's theorem (Theorem 4.1 (4)), $R = \beta X$. Applying Lemma 3, we obtain

$$\tau = w(\beta X) = \prod_{k=1}^{\infty} wU_k,$$

and by Lemma 2, $\tau^{\aleph_0} = \tau$, which completely proves the theorem.

Corollary. *Let*

$$\tau = \sum_{k=1}^{\infty} \mathfrak{m}_k, \quad \mathfrak{m}_k \geq \aleph_0 \quad \text{and} \quad \mathfrak{m}_k < \tau.$$

Then there does not exist an extremally disconnected bicompactum of weight τ .

By virtue of the fact that an infinite extremally disconnected bicompactum is the Stone space of representation of some complete Boolean algebra and conversely, we obtain.

Theorem 4. *The cardinality of every complete infinite Boolean algebra is an admissible cardinal number.*

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- * The set T may also be empty.
- ** If $|T| \geq \aleph_0$, then set $[T] = U_0$; if $|T| < \aleph_0$, then this cardinality has no effect on R .
- Note: Figure translations are in progress. See original paper for figures.*
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