

# A PRIORI ESTIMATES FOR THE RADII OF CURVATURE OF A CONVEX SURFACE

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1967

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**Abstract**

**Full Text**

UDC 513.73

*MATHEMATICS*

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## A PRIORI ESTIMATES FOR THE RADII OF CURVATURE OF A CONVEX SURFACE

In the present note we consider the question of a priori estimates for the principal radii of curvature of a closed convex surface  $F$  satisfying the condition

$$f(R_1, R_2) = \varphi(n), \quad R_1 \geq R_2, \quad (1)$$

where  $R_1, R_2$  are the radii of curvature, and  $n$  is the unit vector of the outer normal to the surface. Both functions  $f$  and  $\varphi$  are twice differentiable. The function  $f$  is assumed to be strictly monotone in both variables, i.e.

$$\partial f / \partial R_1 > 0, \quad \partial f / \partial R_2 > 0. \quad (2)$$

**Theorem 1.** *At the point  $P_1$  of the surface  $F$  where  $R_1$  attains its maximum,*

$$(R_2 - R_1) \frac{\partial f}{\partial R_2} + \frac{\partial^2 f}{\partial R_2^2} \frac{\varphi_s^2}{(\partial f / \partial R_2)^2} \geq \varphi_{ss}.$$

*At the point  $P_2$  of the surface where  $R_2$  attains its minimum,*

$$(R_1 - R_2) \frac{\partial f}{\partial R_1} + \frac{\partial^2 f}{\partial R_1^2} \frac{\varphi_s^2}{(\partial f / \partial R_1)^2} \leq \varphi_{ss}.$$

*The differentiation of  $\varphi$  is performed along the arc  $s$  of the great circle in the corresponding principal direction.*

We prove the first part of the theorem. Suppose that at the point  $P_1$ ,  $R_1 > R_2$ . Then in a neighborhood of this point

$$f(R_1, R_2) \equiv g(R_1 R_2, R_1 + R_2),$$

where  $g$  is also a twice differentiable function. Thus equation (1), in a neighborhood of the point  $P_1$ , can be written in the form

$$g(R_1 R_2, R_1 + R_2) = \varphi(n). \quad (3)$$

Take the point  $P_1$  of the surface as the origin of coordinates, and the principal directions on the surface at this point as the directions of the coordinate axes  $x, y$ . Let, for definiteness, the  $x$ -axis correspond to the principal direction with radius of curvature  $R_1$ . Denote by  $H(x, y, z)$  the support function of the surface. Then, for  $x = y = 0, z = 1$ , we shall have

$$H = 0, \quad H_x = 0, \quad H_y = 0.$$

Accordingly, for the function  $h(x, y) = H(x, y, 1)$ ,

$$h = 0, \quad h_x = 0, \quad h_y = 0.$$

The sum and product of the principal radii of curvature of the surface  $F$  are expressed in the known manner through the second derivatives of the function  $H$  on the unit sphere<sup>1</sup>. If, in these expressions, we pass to the function  $h(x, y)$ , we obtain the values

$$\begin{aligned} R_1 R_2 &= (rt - s^2)(1 + x^2 + y^2), \\ R_1 + R_2 &= ((1 + x^2)r - 2xys + (1 + y^2)t)\sqrt{1 + x^2 + y^2}, \end{aligned} \quad (4)$$

where  $r, s, t$  are the generally accepted notations for the second derivatives of the function

$h$ . Let us note that at the point  $P_1$  itself, i.e., for  $x = y = 0, r = R_1, s = 0, t = R_2$ .

The Cartesian coordinate net on the plane  $z = 1$ , under projection from the point  $P_1$  onto the unit sphere  $x^2 + y^2 + z^2 = 1$ , becomes a curvilinear net. The coordinate lines  $x = \text{const}$  and  $y = \text{const}$  on the sphere are great circles. Draw the cylinder  $Z$  projecting the surface  $F$  onto the plane of the great circle  $y = \text{const}$ . The radius of curvature  $R$  of this cylinder along the line of contact with the surface is contained between the principal radii of curvature of the surface, i.e.,

$$R_2 \leq R \leq R_1. \quad (5)$$

For the radius of curvature  $R$  we have the known expression

$$R = p + p_{ss},$$

where  $p$  is the value of the support function  $H$  on the unit circle  $y = \text{const}$ , and the differentiation is performed with respect to the arc of this circle. Denote

the radius of curvature  $R$ , as a function of the coordinates  $x, y$ , by  $w(x, y)$ . Observing that

$$p = \frac{1}{\sqrt{1+x^2+y^2}} h(x, y),$$

we find for the function  $w(x, y)$  the expression

$$w = r(1+x^2+y^2)^{3/2}/(1+y^2).$$

Since the direction  $y = 0$  at the point  $(0, 0)$  corresponds to the principal direction  $R_1$ , we have  $w(0, 0) = R_1$ , and consequently, in view of inequality (5), the function  $w(x, y)$  attains a maximum at the point  $(0, 0)$ . Therefore, at the point  $(0, 0)$ ,

$$\begin{aligned} w_x &= r_x = 0, & w_y &= r_y = 0, \\ w_{xx} &= r_{xx} + 3r \leq 0, & w_{yy} &= r_{yy} + r \leq 0. \end{aligned}$$

Differentiating equality (3) with respect to  $x$  at the point  $(0, 0)$ , we successively obtain

$$\begin{aligned} g_1(rt + rt_x) + g_2(r_x + t_x) &= \varphi_x, \\ g_1(r_{xx}t + 2r_x t_x + rt_{xx} + 4rt) + g_2(r_{xx} + t_{xx} + 3r + t) + g_{11}(r_{xt} + rt_x)^2 \\ &+ 2g_{12}(r_{xt} + rt_x)(r_x + t_x) + g_{22}(r_x + t_x)^2 = \varphi_{xx}. \end{aligned}$$

From the first equality, noting that at the point  $(0, 0)$ ,  $r_x = 0$ ,  $r = R_1$ ,  $t = R_2$ , we find

$$t_x = \frac{\varphi_x}{g_1 R_1 + g_2} = \varphi_x \left/ \frac{\partial f}{\partial R_2} \right.$$

Accordingly the second equality is transformed into the form

$$w_{xx} \frac{\partial f}{\partial R_1} + w_{yy} \frac{\partial f}{\partial R_2} + (R_2 - R_1) \frac{\partial f}{\partial R_2} + \frac{\partial^2 f}{\partial R_2^2} \frac{\varphi_x^2}{(\partial f / \partial R_2)^2} = \varphi_{xx}.$$

Hence, since  $w_{xx} \leq 0$ ,  $w_{yy} \leq 0$ , we obtain the inequality

$$(R_2 - R_1) \frac{\partial f}{\partial R_2} + \frac{\partial^2 f}{\partial R_2^2} \frac{\varphi_x^2}{(\partial f / \partial R_2)^2} \geq \varphi_{xx}.$$

It is not difficult to verify that

$$\partial x / \partial s = (1 + x^2 + y^2) / \sqrt{1 + y^2}.$$

Therefore at the point  $(0, 0)$ ,  $\varphi_x = \varphi_s$ ,  $\varphi_{xx} = \varphi_{ss}$ , and we obtain the inequality asserted by the theorem.

At the beginning of the proof we excluded the equality  $R_1 = R_2$ , requiring that at the point  $P_1$ ,  $R_1 > R_2$ . But the inequality obtained is, obviously, also valid in

when  $R_1 = R_2$ . In this case it expresses the trivial fact that the function  $\varphi$  attains a maximum at the point  $P_1$ . The proof of the second part of the theorem on the minimum of  $R_2$  is analogous, and therefore we omit it.

Let us consider two examples. Let  $f(R_1, R_2) = R_1 R_2$ . Then our inequality takes the form

$$(R_2 - R_1)R_1 \geq \varphi_{ss}.$$

Hence, observing that  $R_1 R_2 = \varphi$ , we obtain

$$R_1^2 \leq \varphi - \varphi_{ss}.$$

This estimate plays an important role in the solution of the well-known Minkowski problem on the existence of a closed convex surface with given Gaussian curvature as a function of the normal to the surface.

Let  $f(R_1, R_2) = R_1 + R_2$ . Then at the point of minimum of  $R_2$  we shall have

$$(R_1 - R_2) \leq \varphi_{ss}.$$

Hence one obtains the lower estimate for  $R_2$

$$2R_2 \geq \varphi - \varphi_{ss}.$$

This estimate is used in the solution of Christoffel's problem on the existence of a closed convex surface with a prescribed sum of the principal radii of curvature <sup>2</sup>.

From Theorem 1, as a corollary, one obtains the well-known theorem of A. D. Aleksandrov stating that if on a closed convex surface  $f(R_1, R_2) = \text{const}$ , then it is a sphere. Indeed, if this surface is not a sphere, then at the point  $P_1$ ,  $R_1 > R_2$ . On the other hand, by Theorem 1, at this point  $(R_2 - R_1)\partial f/\partial R_2 \geq 0$ , i.e.  $R_2 \geq R_1$ , which is impossible.

From Theorem 1 one can obtain a general condition for the existence of a priori estimates for the radii of curvature of a closed convex surface satisfying condition (1). We shall say that for the functions  $f$  and  $\varphi$  condition (\*) is fulfilled if

$$\lim_{\substack{R_2=R_2(R_1, n) \\ R_1 \rightarrow \infty}} \left\{ (R_2 - R_1) \frac{\partial f}{\partial R_2} + \frac{\partial^2 f}{\partial R_2^2} \frac{\varphi_s^2}{(\partial f/\partial R_2)^2} \right\} < \varphi_{ss}. \quad (*)$$

We shall say that for these functions condition (\*\*) is fulfilled if

$$\lim_{\substack{R_1=R_1(R_2,n) \\ R_2 \rightarrow \infty}} \left\{ (R_1 - R_2) \frac{\partial f}{\partial R_1} + \frac{\partial^2 f}{\partial R_1^2} \frac{\varphi_s^2}{(\partial f / \partial R_2)^2} \right\} < \varphi_{ss}. \quad (**)$$

In these conditions  $R_1(R_2, n)$  and  $R_2(R_1, n)$  are the solutions of equation (1) with respect to  $R_1$  and  $R_2$ , respectively.

**Theorem 2.** *If condition (\*) is fulfilled for the functions  $f$  and  $\varphi$ , then for the radii of normal curvature of the closed convex surface  $F$  satisfying equation (1) there exists an a priori estimate from above. If, however, condition (\*\*) is fulfilled for the functions  $f$  and  $\varphi$ , then for the radii of normal curvature there exists a positive estimate from below. These estimates depend only on the functions  $f$  and  $\varphi$ .*

In conclusion we note that the method presented for obtaining estimates is also applicable to the more general case when the function  $f$  also depends on  $n$  and equation (1) has the form

$$f(R_1, R_2, n) = \varphi(n).$$

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Received  
16 I 1967

## CITED LITERATURE

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*Note: Figure translations are in progress. See original paper for figures.*

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