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Abstract

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Hydromechanics

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KINETIC EQUATIONS FOR THE VORTEX FIELD

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1. In constructing a theory of turbulence, it is convenient to take as the basic quantity a local characteristic of the motion—the velocity vortex $\vec{\Omega} = \text{rot } \mathbf{v}$. The statistical regime for a region of the flow sufficiently far from the walls is determined by the balance between the stretching of vortex filaments and viscous smoothing. The interaction of these two effects was considered in ⁽¹⁾ for small-scale motions.

In the present work, on the basis of the equations of hydrodynamics of an incompressible viscous fluid, equations are obtained for the n -particle distribution densities of the vortex field in the Lagrangian description and for the n -point densities in the Eulerian description. These equations resemble the chain of kinetic equations in statistical physics ⁽²⁾; however, in contrast to ⁽²⁾, the equations are derived for a field. Further, closed equations are obtained for the n -point distribution densities of the vortex field in a locally homogeneous ⁽³⁾ flow at large Reynolds numbers. The role of the “temperature of vortices” is then played by the mean value of the dissipation of kinetic energy $\langle \varepsilon \rangle$.

2. We write the equations for the velocity vortex in an incompressible viscous fluid in the form

$$\begin{aligned} \partial \Omega_i / \partial t + v_k \partial \Omega_i / \partial x_k &= D_{ik} \Omega_k + \nu \Delta \Omega_i, \\ D_{ik} &= \frac{1}{2} (\partial v_i / \partial x_k + \partial v_k / \partial x_i), \end{aligned} \quad (1)$$

where D_{ik} is the tensor of rates of deformation of the fluid particles, ν is the kinematic viscosity, and summation over repeated indices is carried out from 1 to 3.

Consider a volume V bounded by immobile walls. Taking into account the no-slip condition for a viscous fluid on the boundary, we have

$$v_i = \int_V \alpha_i(\mathbf{r}) \Omega'_j d^3 x', \quad \mathbf{r} = \mathbf{x} - \mathbf{x}', \quad \alpha_{ij} = \frac{1}{4\pi} \varepsilon_{ijk} \frac{r_k}{r^3}, \quad (2)$$

$$D_{ik} = \int_V \beta_{ijk}(\mathbf{r}) \Omega'_j d^3 x', \quad \beta_{ijk} = -\frac{3}{8\pi} \frac{\varepsilon_{ijl} r_l r_k + \varepsilon_{kjl} r_l r_i}{r^5}. \quad (3)$$

(ε_{ijk} is the unit axial tensor.) The bar over the integral in (3) means that one must compute the limit, as $\delta \rightarrow 0$, of the integral over the volume V with the ball $r < \delta$ subtracted (principal value).^{*} To (1)–(3) one must add the condition of solenoidality of the vortex, $\text{div } \vec{\Omega} = 0$. However, it follows from (1)–(3) that if this requirement is satisfied at the initial instant of time, it will also be satisfied subsequently.

Passing to the Lagrangian description, we use the condition of incompressibility of the fluid in the form

$$D\vec{\xi}/Da = 1, \quad (4)$$

^{*} The limit certainly exists if the vortex field has continuous first derivatives. This imposes no additional restrictions, since in using equations (1) the existence of second derivatives is assumed.

where the left-hand side is the Jacobian of the transformation of the Lagrangian coordinates $\vec{\xi}(t, \mathbf{a})$, and $\mathbf{a}$ is the initial position of the particle. Taking (4) into account, from (1)–(3) we obtain

$$\frac{d\xi_i}{dt} = \int_V \alpha_i(\vec{\rho}) \Omega'_j d^3 a', \quad \vec{\rho} = \vec{\xi}(t, \mathbf{a}) - \vec{\xi}'(t, \mathbf{a}'); \quad (5)$$

$$\frac{d\Omega_i}{dt} = \Omega_k \int_V^* \beta_{ijk}(\vec{\rho}) \Omega'_j d^3 a' + \nu \Delta \Omega_i. \quad (6)$$

The asterisk over the integral in (6) indicates that the limit as $\delta \rightarrow 0$ is taken of the integral over V with the region $\rho < \delta$ omitted. In the second term on the right in (6) a notation has been used indicating that the Laplacian with respect to $\mathbf{x}$ is to be taken, with the subsequent substitution $\mathbf{x} = \vec{\xi}(t, \mathbf{a})$.

3. We denote the joint probability density of the values of the coordinates and vorticity for n fluid particles $\mathbf{x}^{(1)}, \vec{\omega}^{(1)}, \dots, \mathbf{x}^{(n)}, \vec{\omega}^{(n)}$ at time t , under the condition that at the initial time they occupied the positions $\mathbf{a}^{(1)}, \dots, \mathbf{a}^{(n)}$, by

$$P_n(t, \mathbf{x}^{(1)}, \vec{\omega}^{(1)}, \dots, \mathbf{x}^{(n)}, \vec{\omega}^{(n)} \mid \mathbf{a}^{(1)}, \dots, \mathbf{a}^{(n)}).$$

Using (4), one can show that the expression

$$\int_V \dots \int_V P_n d^3 a^{(1)} \dots d^3 a^{(n)} = F_n(t, \mathbf{x}^{(1)}, \vec{\omega}^{(1)}, \dots, \mathbf{x}^{(n)}, \vec{\omega}^{(n)}) \quad (7)$$

is the n -point distribution density of the vorticity field (i.e., the Eulerian characteristic).

We shall assume that for an initial vorticity field belonging to some set $\mathcal{A}_0$, equations (1), with the condition that the velocity vanish on the boundary, have a unique solution.* If the initial probability distribution is concentrated on some subset of the set $\mathcal{A}_0$, then from (5), (6) one can obtain the following equation for the distribution densities:

$$\frac{\partial P_n}{\partial t} = \sum_{s=1}^n \int_V \mathcal{L}_j^{(s)} Q_{j,n+1} d^3 a^{(n+1)} \quad (n = 1, 2, \dots), \quad Q_{j,n+1} = \int \omega_j^{(n+1)} P_{n+1} d^3 \omega^{(n+1)}; \quad (8)$$

$$\mathcal{L}_j^{(s)} Q_{j,n+1} = -\frac{\partial}{\partial x_i^{(s)}} \int_V \alpha_{ij}(\mathbf{r}^{(s)}) Q_{j,n+1} d^3 x^{(n+1)} - \frac{\partial}{\partial \omega_i^{(s)}} \left[\omega_k^{(s)} \int_V \beta_{ijk}(\mathbf{r}^{(s)}) Q_{j,n+1} d^3 x^{(n+1)} \right] - \nu \frac{\partial}{\partial \omega_j^{(s)}} [\Delta_{x^{(n+1)}} Q_{j,n+1}]$$

Integrating (8) with respect to the initial coordinates of the particles and taking (7) into account, we have

$$\frac{\partial F_n}{\partial t} = \sum_{s=1}^n \mathcal{L}_j^{(s)} \Phi_{j,n+1} \quad (n = 1, 2, \dots), \quad \Phi_{j,n+1} = \int \omega_j^{(n+1)} F_{n+1} d^3 \omega^{(n+1)}. \quad (10)$$

The last equations can also be obtained directly from (1)–(3). Equations (8)–(10) are not difficult to rewrite in terms of the corresponding characteristic functions (see, in particular, (15)).

*

Unfortunately, for three-dimensional flow a theorem of this kind has been proved only “in the small” (4). For the proof of the theorem on the uniqueness of solvability “in the whole,” the results presented in Sec. 2 may be useful; the formulation of the problem in terms of the vorticity field is natural, since in this case the principal three-dimensional effect—the stretching of vortex lines—is singled out.

4. Let us consider a homogeneous flow, passing to the limit $V \rightarrow \infty$. We have

$$F_1 \rightarrow f_1(t, \vec{\omega}), \quad F_2 \rightarrow f_2(t, \mathbf{r}, \vec{\omega}, \vec{\omega}') \quad (\mathbf{r} = \mathbf{x} - \mathbf{x}') \quad \text{etc.}$$

We shall restrict ourselves to the case in which the mean value of the vorticity is zero:

$$\int \omega_i f_1 d^3 \omega = 0. \quad (11)$$

The ratio of the term standing on the left-hand side of (10) to any of the terms on the right-hand side (with the exception of the first term in the first of equations (10), which vanishes by virtue of homogeneity) is, in order of magnitude, equal to $m = (T\Omega)^{-1}$, where T is the characteristic decay time of turbulence and Ω is the root-mean-square value of the vorticity. It is not difficult to see that $m \rightarrow 0$ when $\text{Re} \rightarrow \infty$.^{*} Thus, for large Re we have:

$$\sum_{s=1}^n l_j^{(s)} \varphi_{j,n+1} = 0 \quad (n = 1, 2, \dots), \quad \varphi_{j,n+1} \equiv \int \omega_j^{(n+1)} f_{n+1} d^3 \omega^{(n+1)}, \quad (12)$$

where $l_j^{(s)}$ is the operator obtained from $\mathcal{L}_j^{(s)}$ as $V \rightarrow \infty$. An analogous estimate shows that, for large Re , the same equation will be valid for a locally homogeneous flow. In contrast to (7), we do not introduce here energy sources explicitly, taking into account the locality of the vorticity field. The condition for maintaining the vorticity field at a given “temperature” is written in the form

$$\nu \int \omega_i^2 f_1 d^3 \omega = \langle \varepsilon \rangle. \quad (13)$$

For the stationary problem one must add also the requirement of solenoidality of the vorticity,

$$\partial \varphi_{j,n+1} / \partial x_j^{(n+1)} = 0. \quad (14)$$

The first of equations (12) and condition (14) can be rewritten in the form:

$$\begin{aligned} \theta_i \frac{\partial}{\partial \theta_k} \int \gamma_{ijk} \chi_{j,2} d^3 p &= i \nu \theta_j \int p^2 \chi_{j,2} d^3 p, \quad p_j \chi_{j,2} = 0, \\ \chi_{j,2}(p, \vec{\theta}) &= \frac{1}{(2\pi)^3} \int e^{-i\mathbf{p}\mathbf{r}} \left[\int e^{i\vec{\theta}\vec{\omega}} \varphi_{j,2}(\mathbf{r}, \vec{\omega}) d^3 \omega \right] d^3 r, \end{aligned} \quad (15)$$

$$2\gamma_{ijk} = \varepsilon_{ijl} n_l n_k + \varepsilon_{kjl} n_l n_i, \quad n_i = p_i p^{-1}.$$

The spectral energy density of a locally isotropic turbulent flow $E(p)$ and the characteristic function corresponding to the one-point distribution of the vorticity field $\psi_1(\theta)$ are simply expressed through $\chi_{j,2}$:

$$E = -i2\pi \left. \frac{\partial \chi_{j,2}}{\partial \theta_j} \right|_{\theta=0}, \quad \frac{\partial \psi_1}{\partial \theta} = i\tau_j \int \chi_{j,2} d^3 p, \quad \tau_j = \theta_j \theta^{-1}. \quad (16)$$

For a Gaussian two-point distribution, the formula holds

$$\chi_{j,2} = \frac{i\theta}{4\pi} \psi_1 E(\tau_j - \mu n_j), \quad \mu = \tau_k n_k. \quad (17)$$

Substituting (17) into (15), it is not difficult to verify that the solution cannot be Gaussian.

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* If we put $T \sim v^2 \langle \varepsilon \rangle^{-1}$ (v is the root-mean-square value of the velocity) and use the well-known expression ⁽³⁾ $\langle \varepsilon \rangle \sim v^3 L^{-1}$ (L is the scale of the velocity correlation), then we obtain $m \sim \text{Re}^{-1/2}$ ($\text{Re} = vL\nu^{-1}$). The same small parameter arises in the Lagrangian description of turbulence by the method of random forces ⁽⁵⁾. Direct measurement of the individual terms in the balance equation for the mean square of the vorticity, performed in a wind tunnel behind a grid ⁽⁶⁾, gives the value $m \text{Re}^{1/2} \approx 2$.

Note: Figure translations are in progress. See original paper for figures.

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