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Abstract

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MATHEMATICS

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ON A LEMMA OF THE CALCULUS OF VARIATIONS AND ITS APPLICATION TO EMBEDDING THEOREMS *

(Presented by Academician G. I. Petrov on 6 December 1966)

In the literature ^(1,2) there are estimates for the growth of solutions of equations of elliptic type near the boundary of the domain in which they are prescribed, obtained in the metric C . In this connection let us also note the monograph ⁽³⁾. S. M. Nikol'skii ⁽⁴⁾ obtained other, order-sharp estimates for the growth of a harmonic function and its derivatives near the boundary of the domain in the sense of L_p .

In papers ^(5,6), estimates near the boundary of the domain for partial derivatives of solutions of a linear equation of elliptic type of arbitrary order with constant as well as variable coefficients in the L_2 metric were obtained, sharp and close to the estimates of ⁽⁴⁾; however, the estimates were made, in the case of an equation of order $2l$, for all derivatives starting with the l -th, through the sum of the norms of the derivatives starting with the $(l-1)$ -st and ending with the zeroth. The aim of the present article is to remove the indicated restrictions. Namely, an estimate will be obtained for the norms of any derivative, independently of the order of the equation, through the norm of the solution itself. The result obtained is definitive and cannot be improved in the sense of the degree of the weight function entering into the norms.

§ 1. **Theorem 1.** *Suppose it is known that, for some function u and its generalized derivatives, the inequality*

$$\|u\|_{W_{2,-s}^{(s)}(g)}^2 \leq C_{l,s} \sum_{j=0}^{l-1} \|u\|_{W_{2,-j}^{(j)}(g)}^2, \quad s = l, l+1, \dots, \quad W_{2,-0}^{(0)} = L_2; \quad (1)$$

$$\|u\|_{W_{2,-k}^{(k)}(g)}^2 = \int_g \sum_h \left(\frac{\partial^k u}{\partial x_1^{k_1} \partial x_2^{k_2} \dots \partial x_n^{k_n}} \right)^2 t^{2k} dg,$$

where the sum is extended over all partial derivatives of order k , and t is the distance from the point of integration to the boundary Γ of the domain g ; Γ is assumed sufficiently smooth, and $C_{l,s} > 0$ is a constant independent of both u and t . Then the inequality

$$\|u\|_{W_{2,-k}^{(k)}(g)}^2 \leq C_{l,k} \|u\|_{L_2(g)}^2 \quad (2)$$

holds for all $k = 1, 2, \dots$

In the article ⁽⁷⁾ it was proved that inequality (1) holds for generalized derivatives of a solution v of a linear homogeneous equation of elliptic type of arbitrary order with variable coefficients. Therefore, from Theorem 1 it follows, in particular, that estimate (2) also holds for the aforementioned solution v .

* The result of this article was reported at the Steklov Mathematical Institute at the seminar on function theory and at the World Mathematical Congress in Moscow in August 1966.

§ 2. Theorem 1 is proved with the aid of the following lemmas. Denote

$$I_s^2 = \|u\|_{W_{2,-s}^{(s)}(g)}^2.$$

Lemma 1. For any $s = 1, 2, \dots$ the inequality

$$I_s^2 \leq C_s (I_{s-1}^2 + I_{s-1} I_{s+1}), \quad (3)$$

holds, where $C_s > 0$ is a constant depending on s and on the domain g (for space dimension $n \geq 2$).

Lemma 2. Suppose it is given that

$$x^2 \leq C \left(1 + \sum_{i=1}^q x^{\alpha_i} y^{\beta_i} \right), \quad (4)$$

where $x, y > 0$, $\alpha_i, \beta_i \geq 0$; $\alpha_i + \beta_i < 2$; $C > 0$ is a constant; $q > 0$ is an integer. Then the inequality

$$x^2 \leq \tilde{C}(1 + y^\sigma), \quad (5)$$

holds, where $\tilde{C} > 0$ is a constant depending only on C, q, α_i, β_i , but not on x, y , and

$$\sigma = \max_{(i)} \frac{2\beta_i}{2 - \alpha_i} < 2.$$

Lemma 3. Suppose that for some function u the inequality

$$I_s^2 \leq c_1 \sum_{j=0}^{l-1} I_j^2. \quad (1')$$

holds. Then the inequality

$$I_s^2 \leq \bar{c} I_0^2, \quad (6)$$

is valid, where $\bar{c}$ is a constant depending on c_1 and l .

This lemma is proved with the aid of Lemmas 1 and 2; namely, by induction one proves the inequalities

$$I_k^2 \leq a_k (I_0^2 + I_0^{2-\sigma_k} I_{k+1}^{\sigma_k}); \quad (7)$$

$$I_k^2 \leq b_k (I_0^2 + I_0^{2-\tau_k} I_l^{\tau_k}), \quad (8)$$

where $0 < \sigma_k < 2$; $a_k > 0$; $0 < \tau_k < 2$; $b_k > 0$ are constants, $k = 0, 1, 2, \dots, s-1$.

From inequalities (8) and (1') the inequality (6) follows.

Lemma 4. If the inequality (1') is satisfied, then the relation

$$I_p^2 \leq c' I_0^2 \quad (p = 1, 2, \dots, s-1) \quad (9)$$

holds, where $c' > 0$ is a constant.

Theorem 1 follows from Lemmas 3 and 4.

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