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Abstract

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MATHEMATICS

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ON COMPLETELY REDUCIBLE REPRESENTATIONS OF A SEMISIMPLE LIE ALGEBRA

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A linear representation in a linear space $\mathfrak{M}$ is called **completely reducible** if, for every subspace invariant with respect to the operators of the representation, there exists a complementary invariant subspace. As is well known, a representation of a semisimple Lie algebra $\mathfrak{G}$ of characteristic 0 is completely reducible if the representation space is finite-dimensional. In the infinite-dimensional case there arise representations that are not completely reducible, as is already shown by the example of the Lorentz group. We shall consider some conditions for complete reducibility of representations of the algebra $\mathfrak{G}$.

1. Let us turn to the proof of complete reducibility of representations of the algebra $\mathfrak{G}$ in the finite-dimensional case ⁽¹⁾, Ch. 3, § 7). It is divided into three stages.

A. Let $\mathfrak{M}$ be a finite-dimensional linear space in which a representation of the algebra $\mathfrak{G}$ is given; let $\mathfrak{N}$ be an invariant subspace; E_1 an arbitrary projection onto $\mathfrak{N}$; $\mathfrak{X}$ the set of all linear transformations of $\mathfrak{M}$ that carry $\mathfrak{M}$ into $\mathfrak{N}$, and $\mathfrak{N}$ into $\{0\}$. The space $\mathfrak{N}$ has an invariant complement if there exists a transformation $D \in \mathfrak{X}$ such that, for all $a \in \mathfrak{G}$,

$$[X(a), E_1] = [X(a), D],$$

where $X(a)$ is the representation operator corresponding to $a \in \mathfrak{G}$, and $[X, Y] = XY - YX$.

B (Whitehead's lemma). Let $a \rightarrow f(a)$ be a linear mapping of the algebra $\mathfrak{G}$ into the space $\mathfrak{M}$ such that

$$f([a, b]) = -X(b)f(a) + X(a)f(b), \quad (1)$$

where $[a, b]$ is the commutator in $\mathfrak{G}$. Then, for some $d \in \mathfrak{M}$,

$$f(a) = X(a)d. \quad (2)$$

C. Define a representation $a \rightarrow \tilde{X}(a)$ of the algebra $\mathfrak{G}$ in the space $\mathfrak{X}$ by the formula

$$\tilde{X}(a)P = [X(a), P]$$

for all $P \in \mathfrak{X}$.

If $f(a) = [X(a), E]$, then condition (1) is satisfied, where $X(e)$ is replaced by $\widetilde{X}(e)$, $e \in \mathfrak{G}$. According to B, there exists $D \in \mathfrak{X}$ such that

$$f(a) = [X(a), D].$$

As follows from proposition A, the subspace $\mathfrak{N}$ has an invariant complement in $\mathfrak{M}$.

2. It is easy to see that the arguments in A and C in fact do not depend on the dimension of the space $\mathfrak{M}$. The finite-dimensionality of $\mathfrak{M}$ is essential only in the proof of Whitehead' s lemma. However, in some cases one can dispense with this restriction.

Theorem 1. *Let $\mathfrak{G}$ be a semisimple Lie algebra of characteristic 0; let $a \rightarrow X(a)$ be a linear representation of the algebra $\mathfrak{G}$ in a space $\mathfrak{M}$ of arbitrary dimension. If the Casimir operator G of this representation is invertible and maps $\mathfrak{M}$ onto itself, then the representation is completely reducible.*

Proof. In this case Whitehead' s lemma holds. Indeed, if $a \rightarrow f(a)$ is the above-mentioned mapping of $\mathfrak{G}$ into $\mathfrak{M}$ possessing property (1), then (see ⁽¹⁾, p. 92)

$$X(a)y = Gf(a), \quad (3)$$

where

$$y = \sum_{i=1}^m f(u_i)u^i;$$

$\{u_i\}$ and $\{u^i\}$ are dual bases in $\mathfrak{G}$ with respect to the Killing form; m is the dimension of the algebra $\mathfrak{G}$.

It follows from (3) that $f(a) = X(a)G^{-1}y$, which is what was required to prove.

Theorem 2. *Let $\mathfrak{G}$ be a semisimple Lie algebra over an algebraically closed field of characteristic 0; let $\mathfrak{H}$ be a Cartan subalgebra; let $a \mapsto X(a)$, $a \in \mathfrak{G}$, be a representation of the algebra $\mathfrak{G}$ in a space $\mathfrak{M}$ of arbitrary dimension. If, for all $h \in \mathfrak{H}$, $h \neq 0$, every root $\alpha(h) \neq 0$ of the algebra $\mathfrak{G}$ with respect to the subalgebra $\mathfrak{H}$ is not an eigenvalue of the operator $X(h)$, and $X^{-1}(h)$ maps $\mathfrak{M}$ onto itself, then the representation is completely reducible.*

Proof. As in the preceding case, Weyl' s lemma applies here.

Let l be the rank of the algebra $\mathfrak{G}$; let $\alpha_1(h), \alpha_2(h), \dots, \alpha_l(h)$ be a basis system of roots with respect to the Cartan subalgebra $\mathfrak{H}$. As is known, in the algebra $\mathfrak{G}$ there exists a system of generators consisting of vectors h_i, e_i, f_i , $i = 1, 2, \dots, l$, such that all $h_i \in \mathfrak{H}$ and $[h_i, h_j] = 0$, $[e_i, f_j] = \delta_{ij}h_i$, $[e_j, h_i] = \alpha_j(h_i)e_j$, $[f_j, h_i] = -\alpha_j(h_i)f_j$, where δ_{ij} is the Kronecker symbol, $i, j = 1, 2, \dots, l$. Let $a \mapsto f(a)$,

as before, denote a linear mapping of $\mathfrak{G}$ into $\mathfrak{M}$ possessing property (1). Put $d_j = X^{-1}(h_j)f(h_j)$. According to (1),

$$-X(h_i)f(h_j) + X(h_j)f(h_i) = 0,$$

therefore $d_i = d_j$ for all $i, j = 1, 2, \dots, l$. Moreover,

$$\begin{aligned} \alpha_j(h_i)f(e_j) &= f([e_j, h_i]) = -X(h_i)f(e_j) + X(e_j)f(h_i) = \\ &= -X(h_i)f(e_j) + X(e_j)X(h_i)d_1 = -X(h_i)f(e_j) + X(h_i)X(e_j)d_1 + \\ &\quad + \alpha_j(h_i)X(e_j)d_1 \end{aligned}$$

or

$$[\alpha_j(h_i)E + X(h_i)]f(e_j) = [\alpha_j(h_i)E + X(h_i)]X(e_j)d_1,$$

where E is the identity operator. Since $-\alpha_j(h_i)$ is not an eigenvalue of the operator $X(h_i)$, it follows that $f(e_j) = X(e_j)d_1$. Analogously one proves the equality $f(f_j) = X(f_j)d_1$ for all $j = 1, 2, \dots, l$. Thus, for all vectors p from a certain system of generators of the algebra $\mathfrak{G}$, the condition $f(p) = X(p)d_1$ is fulfilled. In view of (1), it follows from this that this condition is fulfilled for all p , which is what was required to prove.

Remark. In fact, a stronger theorem has been proved. Complete reducibility already holds when $\pm\alpha_j(h_i)$ are not eigenvalues of the operator $X(h_i)$, $j, i = 1, 2, \dots, l$, for at least one j , while the invertibility of the operators $X(h_i)$ for all i and the condition $X^{-1}(h_i)\mathfrak{M} = \mathfrak{M}$ are assumed.

3. Let us apply the results obtained to the consideration of certain representations of the algebra $\mathfrak{G}$.

3a. An important role is played by the so-called e -extremal representations of the algebra $\mathfrak{G}$ ((1), Ch. 7, § 2), which arise in the case when the roots of the algebra belong to the field of its coefficients.

Let, as before, l be the rank of the algebra $\mathfrak{G}$; let $\mathfrak{X}$ be the free associative algebra generated by the generators $x_1, x_2, \dots, x_l$; let $\Lambda \equiv \Lambda(h)$ be a linear function on $\mathfrak{H}$ (see item 2). The representation τ , $a \mapsto X(a)$, $a \in \mathfrak{G}$, is defined by the following conditions:

$$X(h)1 = \Lambda 1, \quad X(e_i)1 = 0, \quad i = 1, 2, \dots, l;$$

$$X(f_i)x_{i_1}x_{i_2}\dots x_{i_r} = x_{i_1}x_{i_2}\dots x_{i_r}x_i,$$

$$X(e_i)x_{i_1}x_{i_2}\dots x_{i_r} = [X(e_i)x_{i_1}x_{i_2}\dots x_{i_{r-1}}]x_{i_r} - \\ -\delta_{i_r,i}(\Lambda - \alpha_{i_1} - \alpha_{i_2} - \dots - \alpha_{i_{r-1}})(h_i)x_{i_1}x_{i_2}\dots x_{i_{r-1}},$$

$$X(h)x_{i_1}x_{i_2}\dots x_{i_r} = (\Lambda - \alpha_{i_1} - \alpha_{i_2} - \dots - \alpha_{i_r})x_{i_1}x_{i_2}\dots x_{i_r},$$

$$h \in \mathfrak{H}.$$

Theorem 3. *The representation τ is irreducible if $\Lambda(h)$ does not belong to the additive group generated by the roots of the algebra $\mathfrak{G}$ with respect to any subalgebra $\mathfrak{H}' \subset \mathfrak{H}$, $\mathfrak{H}' \neq \{0\}$.*

Proof. From the definition it follows that all $X(h)$ are invertible for $h \neq 0$. Since the weight spaces of the algebra $\mathfrak{h}$ in $\mathfrak{r}$ are finite-dimensional, $X^{-1}(h)\mathfrak{r} = \mathfrak{r}$ for $h \neq 0$. Roots are not weights; therefore, by Theorem 2, the representation τ is completely reducible.

Suppose that there exists a proper invariant subspace $\mathfrak{r}_1$ in $\mathfrak{r}$. Let $\mathfrak{r}_2$ be a complementary invariant subspace; $1 = a_1 + a_2$, $a_i \in \mathfrak{r}_i$, $1 \in \mathfrak{r}$, $i = 1, 2$. For some i , $a_i \neq 0$; therefore from the equality $X(h)a_i = \Lambda a_i$ it follows that the vector a_i differs from 1 only by a scalar factor. The representation τ is cyclic with cyclic vector 1. Hence $\mathfrak{r}_i = \mathfrak{r}$, which contradicts the definition of the space $\mathfrak{r}$. The theorem is proved.

3b. Consider some representations of the group $SL(2, R)$. We realize it as the group G_1 of matrices a of the form

$$a = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}$$

with $\alpha\bar{\alpha} - \beta\bar{\beta} = 1$. In the Hilbert space $L^2 = L^2(-\pi, \pi)$ of measurable functions on the unit circle with summable square of the modulus, each complex number $\sigma = i\rho/2$ and j , $j = 0, \frac{1}{2}$, determine a certain representation of the group G_1 of the form ⁽²⁾

$$T_\sigma^j(a)f(\psi) = |\bar{\alpha} - \beta e^{i\psi}|^{-1-2\sigma} e^{ij(\psi' - \psi)} f(\psi'), \quad f(\psi) \in L^2, \quad (4)$$

where $\psi' = a^{-1}\psi$, $e^{i(a\psi)} = (\bar{\alpha}e^{i\psi} + \bar{\beta})/(\beta e^{i\psi} + \alpha)$.

Let $k = 0, \pm 1, \pm 2, \dots$. Finite linear combinations of vectors from the one-dimensional spaces $P_k = \{e^{ik\psi}\}$ form a linear manifold L everywhere dense in L^2 . On the linear manifold L the infinitesimal operators of the representation (4) are defined; moreover the Casimir operator has the form $c(|\rho|^2 - 1)E$, where E is the identity operator and $c = \text{const}$, $c \neq 0$. If $|\rho| \neq 1$, then the conditions of Theorem 1 are satisfied for the representation of the Lie algebra $\mathfrak{G}$ of

the group G_1 defined by the representation (4) of the group G_1 . Any invariant linear manifold $L^{(1)}$ in L (with respect to the representation of the algebra $\mathfrak{G}$) decomposes into a direct sum of the spaces P_k , because P_k is a weight space in the representation (4) with respect to the diagonal subgroup of the group G_1 . Since all the spaces P_k are one-dimensional and orthogonal in L^2 , $L^{(1)}$ and $L^{(2)}$, where $L^{(2)}$ is an invariant complementary linear manifold to $L^{(1)}$, are a pair of orthogonal sets in L^2 . Their closures are invariant subspaces in L^2 with respect to the operators $T_\sigma^j(a)$, and the direct sum of these closures is equal to L^2 . Thus Theorem 4 is proved.

Theorem 4. *The representation $a \mapsto T_\sigma^j(a)$ of the group $SL(2R)$ is completely reducible when $|\rho| \neq 1$.*

In ⁽²⁾ it is proved that these representations are completely irreducible when $\rho \neq \pm i(2n+2j+1)$, $n = 0, \pm 1, \pm 2, \dots$. In the case $\rho = \pm i(2n+2j+1)$, reducible representations arise, each of which contains a finite-dimensional component. From Theorem 2 in ⁽²⁾ and the theorem just proved it follows

Corollary. *The representation $a \mapsto T_\sigma^j(a)$ of the group $SL(2, R)$ for*

$$\rho = \pm i(2n+2j+1), \quad n = 0, \pm 1, \pm 2, \dots, \quad \text{and } |\rho| \neq 1$$

is the direct sum of the representations

$$D_{\frac{|\rho|+1}{2}}^+, \quad D_{\frac{|\rho|+1}{2}}^-$$

and a finite-dimensional representation of dimension $|\rho|$.

3c. Consider the tensor product ν of the representation τ and a finite-dimensional representation of the algebra $\mathfrak{G}$ in 3a.

Theorem 5. *The representation ν of the algebra $\mathfrak{G}$ is completely reducible.*

Proof. It suffices to note that all weights of the representation ν with respect to $\mathfrak{H}$ have the form $\Lambda_0(h) + a_0(h)$, where $\Lambda_0(h)$ is a weight of the representation τ , and $a_0(h)$ is a linear form that is a multiple of the roots of the algebra $\mathfrak{G}$ with respect to $\mathfrak{H}$.

The assertion of Theorem 5 follows from Theorem 4.

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Note: Figure translations are in progress. See original paper for figures.

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