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Abstract

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MATHEMATICS

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ON THE EXTENSION OF MAPPINGS OF TOPOLOGICAL SPACES

(Presented by Academician P. S. Aleksandrov, February 27, 1967)

The paper is devoted to questions concerning the extension of continuous and θ -continuous (see (1)) mappings of topological Hausdorff spaces to their Hausdorff (mainly H -closed) extensions. A space X is called **completely separated** if any two distinct points of it have disjoint closed neighborhoods. A set P in a space X is called δ -closed if, for every point $x \notin P$, there is a neighborhood Ox such that $\langle [Ox] \rangle \cap P = \emptyset$ (everywhere here $[A]$ denotes the closure, and $\langle A \rangle$ the open kernel of the set A) (see (12)). An extension ρX of a space X is called **paracombinatorial** if, for every pair of open disjoint sets in X , the intersection of their closures in ρX lies in X .

Lemma 1. *A Hausdorff extension αX of a Hausdorff space X is H -closed if and only if, for every centered system of sets $\{H_\alpha\}$ open in X , we have*

$$\bigcap_{\alpha} [H_\alpha]_{\alpha X} \neq \emptyset.$$

Theorem 1. *Every perfect continuous mapping f of a Hausdorff space X onto a Hausdorff space Y extends, with preservation of continuity, to a mapping $\hat{f}$ of the Katětov extension τX onto an arbitrary H -closed extension cY of the space Y in such a way that*

$$\hat{f}(\tau X \setminus X) = cY \setminus Y.$$

Proof. Let $\xi \in \tau X \setminus X$, let $\{H'_\alpha\}_{\alpha \in I}$ be the system of all its open neighborhoods in τX , and let $\{H_\alpha\} = \{H'_\alpha \cap X\}$. Then

$$\bigcap_{\alpha} [H_\alpha]_X = \emptyset.$$

In each set H_α choose a point x_α . Let $B_\alpha = f^{-1}fx_\alpha$. The set B_α is bicompat. Denote by $\{O_{\gamma_\alpha} B_\alpha\}_{\gamma_\alpha \in \Gamma_\alpha}$ the system of all open neighborhoods in X of the bicompatum B_α . Construct the systems $\{O_{\gamma_\alpha} B_\alpha\}$ for each $\alpha \in I$. Consider the system

$$\{T_{\gamma_\alpha, \alpha}\} = \{H_\alpha \cup O_{\gamma_\alpha} B_\alpha\},$$

where γ_α runs through Γ_α , and α runs through I . We shall prove that

$$\bigcap_{\gamma_\alpha \in \Gamma_\alpha, \alpha \in I} [T_{\gamma_\alpha, \alpha}]_{\tau X} = \{\xi\}.$$

Let a point $\eta \neq \xi$ be arbitrary, and let $O\eta$ and $O\xi$ be disjoint (open) neighborhoods of the points η and ξ in τX . Let $O\xi \cap X = H_\alpha$. Since B_α is bicomact, the point η has a neighborhood $V\eta$ such that $[V\eta] \cap B_\alpha = \emptyset$. Then

$$V\eta \cap O\eta \cap T_{\gamma_\alpha, \alpha} = \emptyset,$$

where

$$T_{\gamma_\alpha, \alpha} = H_\alpha \cup (X \setminus [V\eta \cap X]_X).$$

Consequently,

$$\eta \notin \bigcap [T_{\gamma_\alpha, \alpha}]_{\tau X} \quad \text{and} \quad \bigcap [T_{\gamma_\alpha, \alpha}]_{\tau X} = \{\xi\}.$$

The set $X \setminus T_{\gamma_\alpha, \alpha}$ is closed in X ; therefore the closed set

$$S_{\gamma_\alpha, \alpha} = f(X \setminus T_{\gamma_\alpha, \alpha})$$

does not cover Y , since $fx_\alpha \notin S_{\gamma_\alpha, \alpha}$. Put

$$L_{\gamma_\alpha, \alpha} = Y \setminus S_{\gamma_\alpha, \alpha}.$$

The family

$$t = \{L_{\gamma_\alpha, \alpha}\}_{\gamma_\alpha \in \Gamma_\alpha, \alpha \in I}$$

of open sets in Y is centered. Indeed, consider an arbitrary finite collection of sets from t :

$$\{L_{\gamma_{\alpha_i}, \alpha_i}\}, \quad i = 1, 2, \dots, n.$$

Let

$$H_\beta = \bigcap_{i=1}^n H_{\alpha_i}.$$

Then

$$fx_\beta \in \bigcap_{i=1}^n L_{\gamma_{\alpha_i}, \alpha_i} \quad \text{and} \quad \bigcap_{i=1}^n L_{\gamma_{\alpha_i}, \alpha_i} \neq \emptyset.$$

From the centeredness of the system $\{L_{\gamma_\alpha, \alpha}\}$ it follows that

$$\bigcap_{\gamma_\alpha, \alpha} [L_{\gamma_\alpha, \alpha}]_{cY} = B \neq \emptyset$$

(Lemma 1). We shall prove,

that the set B reduces to a single point y . Suppose that B contains two points y_1 and y_2 . Let Oy_1 and Oy_2 be their disjoint open neighborhoods. Then

$$f^{-1}(Oy_1 \cap Y) \cap f^{-1}(Oy_2 \cap Y) \neq \emptyset,$$

but

$$T_{\gamma_\alpha, \alpha} \cap f^{-1}(Oy_1 \cap Y) \neq \emptyset \quad \text{and} \quad T_{\gamma_\alpha, \alpha} \cap f^{-1}(Oy_2 \cap Y) \neq \emptyset$$

for arbitrary γ_α and α .

We shall prove that: (A) for any open neighborhood Oy of an arbitrary point $y \in B$ we have

$$\xi \in [f^{-1}(Oy \cap Y)]_{\tau X}.$$

Indeed, suppose the contrary: assume that the point ξ has an open neighborhood $O\xi$ not intersecting $f^{-1}Oy$. Let $O\xi \cap X = H_\alpha$. Then $fx_\alpha \neq y$. Let Vy and Vfx_α be disjoint open neighborhoods of the points y and fx_α in cY . Then, for

$$T_{\gamma_\alpha, \alpha} = H_\alpha \cup f^{-1}(Vfx_\alpha)$$

and

$$Wy = Vy \cap Oy,$$

we would have

$$Wy \cap L_{\gamma_\alpha, \alpha} = \emptyset,$$

which is impossible. Thus (A) is proved. Consequently,

$$\xi \in [T_{\gamma_\alpha, \alpha} \cap f^{-1}(Oy_1 \cap Y)]_{\tau X} \cap [T_{\gamma_\alpha, \alpha} \cap f^{-1}(Oy_2 \cap Y)]_{\tau X},$$

where $T_{\gamma_\alpha, \alpha}$ is chosen arbitrarily. But the sets

$$T_{\gamma_\alpha, \alpha} \cap f^{-1}(Oy_1 \cap Y) = M_1$$

and

$$M_2 = T_{\gamma_\alpha, \alpha} \cap f^{-1}(Oy_2 \cap Y)$$

are open in X and do not intersect. Therefore (since τX is a paracombinatorial extension)

$$[M_1]_{\tau X} \cap [M_2]_{\tau X} \cap (\tau X \setminus X) = \emptyset.$$

We have arrived at a contradiction. Hence $B = \{y\}$.

Put $\hat{f}\xi = y$, and $\hat{f}x = fx$ for points $x \in X$. We obtain a mapping $\hat{f}$ of the space τX into the space cY , which extends f , and

$$\hat{f}(\tau X \setminus X) \subseteq cY \setminus Y.$$

The latter inclusion follows from the fact that the point $\hat{f}\xi = y \in cY \setminus Y$, for for any point $\bar{y} \in Y$ the set $B\bar{y} = f^{-1}\bar{y}$ and the point $\xi \in \tau X \setminus X$ have disjoint open neighborhoods $OB\bar{y}$ and $O\xi$, and then, if $O\xi \cap X = H_\alpha$, we have

$$\bar{y} \notin L_{\gamma_\alpha, \alpha}$$

for some γ_α .

It is clear that the mapping $\hat{f}$ is continuous at each point $x \in X$. Let $\xi \in \tau X \setminus X$, $y = \hat{f}\xi$, and let Oy be an arbitrary open neighborhood of the point y in cY , $T = O_y \cap Y$. Then

$$\xi \in [f^{-1}T]_{\tau X}$$

(see (A)), and since $f^{-1}T$ is open in X , it follows that

$$\{\xi\} \cup f^{-1}T = O\xi$$

is a neighborhood of the point ξ such that

$$\hat{f}O\xi \subseteq Oy.$$

Consequently, $\hat{f}$ is continuous on τX , and since

$$\hat{f}\tau X \supseteq Y,$$

we have

$$\hat{f}\tau X = cY,$$

and

$$\hat{f}(\tau X \setminus X) = cY \setminus Y.$$

The theorem is proved.

Corollary. The Katětov extension τX of a Hausdorff space X is mapped continuously and fixing the points of X onto every H -closed extension cX of the space X , and the remainder $\tau X \setminus X$ is mapped onto the remainder $cX \setminus X$.

Theorem 2. If f is a perfect continuous mapping of a Hausdorff space X onto an H -closed space Y , then X is also H -closed.

Proof. The mapping $f : X \rightarrow Y$ can be extended, preserving continuity, to a mapping

$$\hat{f} : \tau X \rightarrow Y$$

(Theorem 1), but a perfect mapping is absolutely closed (see (2)), which entails the equality

$$\tau X = X.$$

Consequently, the space X is H -closed. The theorem is proved.

Definition. A mapping f of a space X onto a space Y is called δ -continuous if, for every point $x \in X$ and every neighborhood Oy of the point $y = fx \in Y$, there is a neighborhood Ox of the point x such that

$$fOx \subseteq \langle [Oy] \rangle.$$

Lemma 2. A mapping $f : X \rightarrow Y$ is δ -continuous if and only if, for every set H open in Y , the set

$$f^{-1}(\langle [H] \rangle)$$

is open in X .

Theorem 3. A θ -continuous mapping f of a Hausdorff space X onto a Hausdorff space Y is θ -continuous in each of the following cases: (a) the space X is H -closed and completely regular; (b) the mapping f is open.

Proof. (a). The space Y is H -closed and completely regular. Let x be an arbitrary point of X , and let Oy be an arbitrary neighborhood of the point $y = fx$. We shall prove that the point y has a neighborhood Vy such that

$$[Vy] \subseteq \langle [Oy] \rangle.$$

The set

$$S = Y \setminus \langle [Oy] \rangle = [Y \setminus [Oy]]$$

is H -closed as the closure of an open set. For each point $z \in S$

and the point y have open neighborhoods Oz and Ozy such that $[Oz] \cap [Ozy] = \emptyset$. There is a finite number of points z_i , $i = 1, 2, \dots, n$, such that $\bigcup_{i=1}^n [Oz_i] \supseteq S$. Put $Vy = \bigcap_{i=1}^n Ozy_i$. We have $[Vy] \cap S = \emptyset$ (for $[Oz_iy] \cap [Oz_i] = \emptyset$), whence $[Vy] \subseteq \langle [Oy] \rangle$. Let Ox be a neighborhood of the point x such that $fOx \subseteq [Vy]$. Then $fOx \subseteq \langle [Oy] \rangle$. (a) is proved.

(b). The mapping $f : X \rightarrow Y$ is open. Let H be an arbitrary open set in Y . Since for each point $z \in f^{-1}(\langle [H] \rangle) = K$ there is a neighborhood Oz such that $fOz \subseteq [H]$, the set $L = \bigcup_{z \in K} Oz$ will be an open set such that $fL = [H]$, $fL \subseteq \langle [H] \rangle$, $fL \supseteq \langle [H] \rangle$; consequently, $fL = \langle [H] \rangle$, $L = f^{-1}(\langle [H] \rangle)$, and the mapping f is δ -continuous (Lemma 2). (b) is established. The theorem is proved.

Theorem 4. Every open θ -continuous mapping f of a Hausdorff space X onto a Hausdorff space Y can be extended, with preservation of θ -continuity, to a mapping $\hat{f}$ of any paracombinatorial (not necessarily H -closed) extension pX into any H -closed extension cY of the space Y , in such a way that $\hat{f}(pX \setminus X) \subseteq cY \setminus Y$.

Proof. Let $p \in pX \setminus X$ be arbitrary, and let $\{H'_\alpha\}_{\alpha \in I}$ be the system of all its open neighborhoods in pX , $\{H_\alpha\} = \{H'_\alpha \cap X\}$. The family $\{fH_\alpha\}$ of open sets in Y is centered; therefore $B = \bigcap_\alpha [fH_\alpha]_{cY} \neq \emptyset$ (Lemma 1). The set B reduces to a single point η . Suppose the contrary—that B contains two points η_1 and η_2 . Let $O\eta_1$ and $O\eta_2$ be disjoint open neighborhoods of the points η_1 and η_2 . In view of the openness of f , the mapping f is δ -continuous (Theorem 3). Then the sets $L_1 = f^{-1}(\langle [O\eta_1 \cap Y] \rangle)$ and $L_2 = f^{-1}(\langle [O\eta_2 \cap Y] \rangle)$ are open in X and do not intersect. But $[L_1]_{pX} \cap [L_2]_{pX} = K \ni p$, which contradicts the paracombinatoriality of the extension pX . Hence $B = \{\eta\}$.

Put $\hat{f}p = \eta$, $\hat{f}x = fx$ for points $x \in X$. We shall prove that the mapping $\hat{f}$, which is an extension of f , is θ -continuous.

(M). Let the set A be open in X , and let $z \in [A]_{pX} \setminus [A]_X$. Then $\xi = \hat{f}z \in [fA]_{cY}$. Indeed, if the point ξ had an open neighborhood $O\xi$ not intersecting

fA , then the open sets $S = f^{-1}(\langle [O\xi \cap Y] \rangle)$ and A in X would not intersect, but $[S]_{pX} \cap [A]_{pX} = L \ni z \in pX \setminus X$, which contradicts the paracombinatoriality of pX .

Let the point $x \in X$ be arbitrary; $y = \hat{f}x = fx$; let Oy be an arbitrary open neighborhood of the point y ; let Ox be an open neighborhood of the point x in X such that $fOx \subseteq \langle [Oy \cap Y] \rangle$; let $\widehat{Ox}$ be the largest open set in pX that cuts out Ox in X . If $z \in \widehat{Ox} \setminus Ox$, then $z \in [Ox]_{pX}$, whence $\hat{f}z \in [fOx]_{cY} \subseteq [Oy]_{cY}$ (see (M)); consequently, $\widehat{fOx} \subseteq [Oy]_{cY}$, i.e. the mapping $\hat{f}$ is θ -continuous at the point $x \in X$.

Now let $p \in pX \setminus X$, $\eta = \hat{f}p \in cY$, and let $O\eta$ be an arbitrary open neighborhood of the point η in cY . The set $K = f^{-1}(\langle [O\eta \cap Y] \rangle_Y)$ is open in X , and $p \in [K]_{pX}$. Note that $[K]_{pX} \cup [X \setminus [K]]_{pX} = pX$, and, by the paracombinatoriality of the extension pX , the point $p \notin [X \setminus [K]]_{pX}$; therefore the point p has a neighborhood Op such that $Op \subseteq [K]_{pX}$. Let the point $a \in [K]_{pX} \setminus X$ be arbitrary. Then $b = \hat{f}a \in [O\eta]_{cY}$ (see (M)); consequently, $fOp \subseteq [O\eta]_{cY}$, i.e. the mapping $\hat{f}$ is θ -continuous at the point $p \in pX \setminus X$ and, thus, θ -continuous on pX . It is not hard to verify that if $p \in pX \setminus X$, then the point $\eta = \hat{f}p \in cY \setminus Y$, i.e. $\hat{f}(pX \setminus X) \subseteq cY \setminus Y$. The proof is complete.

Theorem 5. *In order that a θ -continuous mapping f of an everywhere dense set S in a space X into an H -closed completely Hausdorff space Y can be extended to all of X with preservation of θ -*

continuity, it is necessary and sufficient that, for any two disjoint canonically closed sets B_1 and B_2 in Y , the preimages $f^{-1}B_1$ and $f^{-1}B_2$ have disjoint closures in X .

Proof of necessity. The θ -continuous mapping $f : S \rightarrow Y$ extends to all of X . The mapping f is δ -continuous (Theorem 3). Let B_1 and B_2 be canonically closed disjoint sets in Y . B_1 and B_2 are δ -closed as closures of open sets (Lemma 2 from (3)). Suppose, for example, that the point $\xi \in [f^{-1}B_1]_X$. Then $f\xi \in [B_1]_\delta = B_1$, by the δ -continuity of f ; therefore $\xi \notin [f^{-1}B_2]_X$ and

$$[f^{-1}B_1] \cap [f^{-1}B_2]_X = \emptyset.$$

Proof of sufficiency. Let a point $x \in X$ be arbitrary, and let $\mathfrak{F}$ be the trace on S of the filter of neighborhoods Bx of the point x . $f(\mathfrak{F})$ is a filter base in Y , and since Y is H -closed, $f(\mathfrak{F})$ has points of δ -adherence (Theorem 2 from (3)). We prove that the set

$$L = \bigcap [fV_\alpha]_\delta$$

reduces to a single point y (where $V_\alpha = S \cap W_\alpha$, and W_α ranges over Bx). Let $y_1 \neq y_2$, $y_1, y_2 \in L$. Choose open neighborhoods Oy_1 and Oy_2 of the points y_1 and y_2 such that $[Oy_1] \cap [Oy_2] = \emptyset$. The sets $[Oy_1]$ and $[Oy_2]$ are canonically closed; hence

$$[f^{-1}([Oy_1])]_X \cap [f^{-1}([Oy_2])]_X = \emptyset.$$

But $V_\alpha \cap f^{-1}([Oy_1]) \neq \emptyset$ and $V_\alpha \cap f^{-1}([Oy_2]) \neq \emptyset$ for every V_α ; consequently the point

$$x \in [f^{-1}([Oy_1])]_X \cap [f^{-1}([Oy_2])].$$

The contradiction obtained shows that the set L consists of a single point y . By Lemma 2 from ⁽⁴⁾, $f(\mathfrak{F})$ δ -converges to the point y (i.e., contains the nuclei of the closures of all neighborhoods of the point y).

Put $\hat{f}x = y$. It is clear that $\hat{f}x = fx$ on the set S . We prove that $\hat{f}$ is a θ -continuous extension of the mapping f to X . Consider an arbitrary open neighborhood Oy of the point $y = fx$. There exists an open neighborhood W_α of the point x such that $fV_\alpha \subseteq [Oy]$, where $V_\alpha = S \cap W_\alpha$. For an arbitrary point $z \in W_\alpha$ we have

$$\hat{f}z \in [fV_\alpha]_\delta = [[Oy]]_\delta = [Oy]$$

(for the image of the trace on S of the filter of neighborhoods of the point z δ -converges to the point $\hat{f}z$), i.e. $\hat{f}W_\alpha \subseteq [Oy]$. The theorem is proved.

Corollary 1 (A. D. Taimanov's theorem). *In order that a continuous mapping f from an everywhere dense subset S of a space X into a bicomact space Y be extendable to all of X with preservation of continuity, it is necessary and sufficient that, for any two canonically closed disjoint sets B_1 and B_2 in Y , the preimages $f^{-1}B_1$ and $f^{-1}B_2$ have disjoint closures in X .*

Corollary 2. *Every continuous perfect mapping of an everywhere dense subset S of a Hausdorff space X into an H -closed completely nonseparable space Y extends to a θ -continuous mapping of the space X into the space Y .*

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