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1967

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Abstract

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UDC 517.11

MATHEMATICS

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SOME RELATIONS BETWEEN PROPERTIES OF CONSTRUCTIVE FUNCTIONS AND OPERATORS FROM QUASI-NUMBERS TO QUASI-NUMBERS

(Presented by Academician A. A. Dorodnitsyn, 12 I 1967)

In this note the definitions and results of ⁽¹⁻⁷⁾ are used. By $\alpha^{(1)}$ we denote an algorithm such that, for any P in $\mathfrak{Ch}_3$ ^{(2), p.77}, $\alpha^{(1)}(P) = P$, if $\diamond$ does not occur in P , and $\alpha^{(1)}(P) = P_1 \diamond$, if $P = P_1 \diamond P_2$ and $\diamond$ does not occur in P_2 .

Let P_1, P_2 be words in $\mathfrak{Ch}_3$. We shall say that P_1 is **duplex-equal** to P_2 and write $P = P_2$, if for all FR -numbers x_1, x_2 , from the conditions $\alpha^{(1)}(x_1) = \alpha^{(1)}(P_1)$, $\alpha^{(1)}(x_2) = \alpha^{(1)}(P_2)$ it follows that $x_1 = x_2$. The relations $P_1 < P_2$, $P_1 \leq P_2$, $P_1 \neq P_2$ are defined analogously.

In ⁽⁷⁾ the concept of an operator from quasi-numbers to quasi-numbers (o.q.n.) was introduced. In an analogous way one can define the concept of an operator from FR -numbers to quasi-numbers and of an operator from quasi- FL -numbers to quasi- FL -numbers (the concept of a quasi- FL -number was introduced in ⁽⁷⁾). The present note is devoted to comparing the properties of such operators with the properties of constructive functions.

Below, the letters n, r, x, p, q , with or without indices, are used respectively to denote natural numbers, rational numbers, FR -, quasi- FL -, and quasi-numbers.

All constructive functions and operators under consideration are assumed to be everywhere defined. Mention of this will often be omitted. The adjective "constructive" will also often be omitted.

1. One of the essential characteristics of constructive functions is their continuity ^(3,4). Operators from quasi-numbers to quasi-numbers also possess certain continuity properties: the author has proved that such operators cannot have constructive discontinuities ⁽⁷⁾. As is known (cf. ⁽²⁾, p. 149, Theorem 6.6.1), operators from FR -numbers to quasi-numbers can have constructive discontinuities. In particular, one can construct an everywhere-defined signum-operator from FR -numbers to quasi-numbers. An analogous theorem can be formulated for operators from quasi- FL -numbers to quasi- FL -numbers.

We shall say that an o.q.n. ψ is **equivalent to a constructive function** φ , if for all q, x , from the condition $q = x$ it follows that $\psi(q) = \varphi(x)$. The equivalence relation between constructive functions and operators of the other two types under consideration is defined analogously.

Theorem 1.1–1.3. *For every function one can construct an equivalent operator from quasi-numbers to quasi-numbers (respectively, an operator from quasi-FL-numbers to quasi-FL-numbers and an operator from FR-numbers to quasi-numbers).*

Theorem 1.3 is obvious.

Theorem 1.1 follows without difficulty from G. S. Tseitin's theorem on the continuity of constructive functions ⁽⁴⁾.

Theorem 1.2 is proved with the aid of the above-mentioned theorem of G. S. Tseitin and the following lemma: one can construct an algorithm β such that for every-

of each quasi-FL-number p , the algorithm $\tilde{\beta}_{p\Box}$ is an FR-number sequence such that

$\neg\neg\exists n (\tilde{\beta}_{p\Box}(n) = p)$. Note that, in view of Theorems 2 and 3 (7), the analogous assertion does not hold for quasinumbers. More precisely, there is no algorithm which, for every quasinumber, constructs a sequence of FR-numbers (quasi-FL-numbers) in which an FR-number (respectively, a quasi-FL-number) duplex-equal to the original quasinumber cannot fail to occur. This strengthens the well-known result of G. S. Tseitin on the impossibility of an algorithm constructing, for every quasinumber, an FR-number duplex-equal to it.

Naturally, the question arises whether Theorems 1.1–1.3 can be reversed. A negative answer to it in the case of operators from FR-numbers to quasinumbers and of operators from quasi-FL-numbers to quasi-FL-numbers plainly follows from the existence of discontinuous operators of this type.

A negative answer to this question in the case of operators from quasinumbers to quasinumbers, and also for operators of the other two types considered that possess certain continuity properties, follows from the results of § 2.

2. Let Ψ be an operator from quasinumbers to quasinumbers.

We shall call an algorithm ρ of type $(\mathbb{N} \rightarrow \mathbb{N})$ a **modulus of continuity** of Ψ at the point q if, for every q_1 , from the condition $|q_1 - q| <_g 2^{-\rho(n)}$ it follows that $|\Psi(q_1) - \Psi(q)| <_g 2^{-n}$.

We shall call an algorithm θ a **weak modulus of continuity** of Ψ at the point q if, for every n , $\theta(n)$, $\theta(n)$ is a quasinumber, moreover $\theta(n) > 0$, and for every q_1 , from the condition $|q_1 - q| <_g \theta(n)$ it follows that

$$|\Psi(q_1) - \Psi(q)| <_g 2^{-n}.$$

We shall say that Ψ is **continuous (weakly continuous)** if one can construct an algorithm γ such that, for every q , the algorithm $\tilde{\gamma}_{q\Box}$ is a modulus of continuity (a weak modulus of continuity) of Ψ at the point q .

We shall call the operator Ψ **uniformly continuous** if one can construct an algorithm ρ of type $(\mathbb{N} \rightarrow \mathbb{N})$ such that, for all n, q_1, q_2 , if $|q_1 - q_2| <_g 2^{-\rho(n)}$, then

$$|\Psi(q_1) - \Psi(q_2)| <_g 2^{-n}.$$

We call the operator Ψ **effectively non-uniformly continuous** if one can construct sequences of quasinumbers $\beta^{(1)}, \beta^{(2)}$ and a natural number n_0 such that, for every n ,

$$|\beta^{(1)}(n) - \beta^{(2)}(n)| <_g 2^{-n} \quad \text{and} \quad |\Psi(\beta^{(1)}(n)) - \Psi(\beta^{(2)}(n))| \geq_g 2^{-n_0}.$$

We shall say that the operator Ψ is **continuous at every duplex point** if one can construct an algorithm γ such that, for every x , $\tilde{\gamma}_{x\Box}$ is a modulus of continuity of Ψ at the point $\alpha^{(1)}(x)$.

In a natural way one may define the notion of an O.C.C. that is continuous, weakly continuous, continuous at every duplex point, uniformly continuous, and effectively non-uniformly continuous on a given segment $x_1\Delta x_2$.

Theorem 2.1. A continuous O.C.C. cannot be effectively non-uniformly continuous.

Corollary 2.1. An O.C.C. continuous on $x_1\Delta x_2$ cannot be effectively non-uniformly continuous on $x_1\Delta x_2$.

The proof of Theorem 2.1 is based on the following simple lemma: whatever the sequence of quasinumbers β may be, there is no algorithm α such that, for every q , $\alpha(q)$ is a natural number, and for every n , if $n \geq_g \alpha(q)$, then $q \neq \beta(n)$. Let us note that the analogous assertion is false for FR -numbers.

Below, by Φ we denote an arbitrarily fixed rational segment exact disjoint covering of $0\Delta 1$ (see (5)). The existence of such coverings is proved in (5).

The following theorem gives a certain criterion for weak continuity and continuity at every duplex point of an o.c.c.

Theorem 2.2. If an o.c.c. Ψ is uniformly continuous on each segment $\Phi(n)$, then: 1) Ψ is weakly continuous on $0\Delta 1$; 2) Ψ is continuous at every duplex point of $0\Delta 1$.

From Theorems 1.1, 2.2, Corollary 2.1, and the results of (5), the following corollary follows.

Corollary 2.2. One can construct an o.c.c. that is weakly continuous on $0\Delta 1$, continuous at every duplex point of $0\Delta 1$, but not continuous on $0\Delta 1$.

The following lemma gives a method for constructing various examples of operators from quasi-numbers to quasi-numbers.

Lemma 2.1. Let F be an algorithm such that, for every n , $\tilde{F}_{n\Box}$ is a function. Then one can construct a sequence of quasi-numbers λ and an everywhere defined o.c.c. Ψ_1 in such a way that, for all n, x, q , if $x \in \Phi(n)$ and $q =_g x$, then

$$\tilde{F}_{n\Box}(x) = \Psi_1(q) + \lambda(n).$$

The following theorem uses certain details of the construction of the sequence λ in the proof of Lemma 2.1 and Theorem 2.2.

Theorem 2.3. One can construct an everywhere defined o.c.c. Ψ_3 such that

- 1) $\forall q(\neg\neg((\Psi_3(q) =_g 1) \vee (\Psi_3(q) =_g -1)))$;
- 2) $\Psi_3(0) =_g -1, \Psi_3(1) =_g 1$;
- 3) Ψ_3 is weakly continuous;
- 4) Ψ_3 is continuous at every duplex point.

Thus, for operators from quasi-numbers to quasi-numbers, Cauchy's theorem is refuted by example. At the same time, as is known (see (6)), a constructive function φ taking exactly two distinct values on $0\Delta 1$ cannot be everywhere defined on $0\Delta 1$. (Moreover, for every such function φ one can construct a quasi-number q such that, if $x =_g q$, then $x \in 0\Delta 1$ and $\neg!\varphi(x)$.)

Thus, Theorem 2.3 establishes an essential difference between the properties of constructive functions and operators from quasi-numbers to quasi-numbers.

Corollary 2.3. One can construct an everywhere defined o.c.c. not equivalent to any constructive function.

This corollary follows directly from Theorem 2.3. From this theorem one also easily obtains the following stronger assertion (as Ψ_4 one may take any o.c.c. satisfying Theorem 2.3).

Theorem 2.4. One can construct an everywhere defined o.c.c. Ψ_4 and an algorithm $\mathfrak{L}$ such that, for any function φ , the following hold:

- 1) $!\mathfrak{L}(\xi\varphi 3)$;
- 2) $!\mathfrak{L}(\xi\varphi 3)$ is an FR -number from $0\Delta 1$;
- 3) if $q =_g \mathfrak{L}(\xi\varphi 3)$, then

$$\Psi_4(q) \neq \varphi(\mathfrak{L}(\xi\varphi 3)).$$

We shall say that a function φ *increases at a point* x if there cannot fail to exist a natural number n_0 such that, for all x_1, x_2 , from the condition

$$x - 2^{-n_0} < x_1 < x < x_2 < x + 2^{-n_0}$$

it follows that

$$\varphi(x_1) < \varphi(x) < \varphi(x_2).$$

An analogous definition is adopted for operators from quasi-numbers to quasi-numbers. One can show that if a function φ increases at every point of the

interval $0\nabla 1$, then φ is an increasing function on $0\Delta 1$ (i.e., for $x_1, x_2 \in 0\Delta 1$ and $x_1 < x_2$, one has $\varphi(x_1) < \varphi(x_2)$).

Theorem 2.5. One can construct an everywhere defined o.c.c. Ψ_5 such that:

- 1) Ψ_5 increases at every point;
- 2) $\Psi_5(0) = 1, \Psi_5(1) = -1$;
- 3) Ψ_5 is weakly continuous;
- 4) Ψ_5 is continuous at every duplex point.

This theorem shows a difference in the properties of constructive

functions and operators from quasi-numbers to quasi-numbers. In particular, Corollary 2.3 also follows from Theorem 2.5.

It follows from Theorem 2.3 that one can construct a continuous operator from FR -numbers to quasi-numbers which takes exactly two values and, consequently, is not equivalent to any constructive function.

The results of this section can be reformulated for operators from quasi- FL -numbers to quasi- FL -numbers.

The author expresses his gratitude to A. A. Markov for his constant attention to the work.

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Received
29 XII 1966

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