

INVARIANT SOLUTIONS OF EQUATIONS DESCRIBING THE MOTION OF A LIQUID AND GAS IN LONG PIPELINES

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1967

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Abstract

Full Text

UDC 532.542

HYDROMECHANICS

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INVARIANT SOLUTIONS OF EQUATIONS DESCRIBING THE MOTION OF A LIQUID AND GAS IN LONG PIPELINES

(Presented by Academician L. I. Sedov, 22 X 1966)

1°. Consider the system of equations

$$\partial u / \partial t + \partial v / \partial x = 0, \quad \partial u / \partial x + v^n / u^m = 0. \quad (1)$$

This system covers all practically important cases of the motion of a liquid and gas in a long pipeline. Thus, $m = 1$, $n = 2$ corresponds to turbulent gas flow, $m = n = 1$ to laminar flow. For $m = 0$, $n = 2$ one obtains the equations of motion of a drop liquid. In calculating gas networks one assumes $m = 0$; $1 < n < 2$ depends on the resistance law.

2°. We shall carry out a group classification of equations (1), using the methods developed by L. V. Ovsyannikov ^(1,2).

We investigate whether the system admits an operator

$$X = \xi \partial / \partial x + \tau \partial / \partial t + \varphi \partial / \partial u + \psi \partial / \partial v. \quad (2)$$

It is necessary that equations (1) be invariant with respect to the first prolongation of the operator X .

Calculations show that, for general values of m and n , the principal group of system (1) is generated by four independent operators

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial t}, \quad X_3 = x \frac{\partial}{\partial x} + \frac{t}{p} \frac{\partial}{\partial t} - \frac{1}{n} v \frac{\partial}{\partial v},$$

$$X_4 = kx \frac{\partial}{\partial x} + u \frac{\partial}{\partial u} + (k+1)v \frac{\partial}{\partial v}.$$

Here $k = (m - n + 1)/(n + 1)$, $p = n/(n + 1)$.

The finite equations of this group contain 4 parameters a, b, c, d

$$x' = (1 + c + kd)x + a, \quad t' = (1 + c/p)t + b,$$

$$u' = (1 + d)u, \quad v' = \{1 - c/n + (k + 1)d\}v.$$

In addition, for $n = 1$, $m = -4/3$ the operator is possible

$$X_5 = -x^2 \partial / \partial x + 3xu \partial / \partial u + (xv - 3u^{-1/3}) \partial / \partial v.$$

A special case occurs for $n = -1$, when the equations admit an infinite group. The components of the operator (2) contain one or two arbitrary functions and take the following values:

$m \neq -2$:

$$\xi = ax + B(t, u), \quad \tau = bt + c, \quad \varphi = \frac{b}{m+2}u,$$

$$\psi = \left(a - \frac{m+1}{m+2}b\right)v + u^{-m} \frac{\partial B}{\partial u};$$

$m = -2$:

$$\xi = A(t, u)x + B(t, u), \quad \tau = bt + c, \quad \varphi = \frac{bt+d}{2}u + \frac{b}{2}u \ln u,$$

$$\psi = \left\{A(t, u) + \frac{t + \ln u - 1}{2}b + \frac{d}{2}\right\}v + u^2x \frac{\partial A}{\partial u} + u^2 \frac{\partial B}{\partial u}.$$

The functions $A(t, u)$ and $B(t, u)$ satisfy the equation

$$\frac{\partial}{\partial t} - \frac{\partial}{\partial u} u^{-m} \frac{\partial}{\partial u} = 0.$$

We do not consider the heat-conduction equation ($m = 0$; $n = 1$) and the trivial case $n = 0$. When $n = 1$, system (1) describes the phenomenon of nonlinear heat conduction, and the results coincide with those presented in [2].

3°. In what follows, bearing in mind the investigation of nonstationary motion in a long pipeline, we shall restrict ourselves to the study of the group with infinitesimal operators $X_1 - X_4$.

Below is written the optimal system of subgroups θ_1 for the case $k \neq 0$ ($m - n + 1 \neq 0$):

$$X_1, X_2, X_1 + X_2, X_4, X_2 + X_4, \alpha X_4 + X_3, X_1 - kX_3 + X_4, \quad (3)$$

where α is an arbitrary constant.

Each subgroup from the collection (3), except the first, makes it possible to reduce system (1) to two ordinary differential equations of the first order.

To the operator X_2 there corresponds a solution of the form

$$u = f(x), \quad v = h(x); \quad (4)$$

to the operator $X_1 + X_2$:

$$u = f(x - t), \quad v = h(x - t); \quad (5)$$

to the operator X_4 :

$$u = x^{1/k} f(t), \quad v = x^{1+1/k} h(t); \quad (6)$$

to the operator $X_2 + X_4$:

$$u = e^t f(\eta), \quad v = e^{(k+1)t} h(\eta), \quad \eta = e^t x^{-1/k}; \quad (7)$$

to the operator $X_3 + \alpha X_4$:

$$u = t^{\alpha p} f(\eta), \quad v = t^{\alpha p(k+1)-1/(n+1)} h(\eta), \quad \eta = t x^{-1/p(1+k\alpha)}; \quad (8)$$

to the operator $X_1 - kX_3 + X_4$:

$$u = t^{n/(n-m-1)} f(\eta), \quad v = t^{(m+1)/(n-m-1)} h(\eta), \quad \eta = e^{-x} t^{n/(n-m-1)}. \quad (9)$$

4°. A solution in the form (4) corresponds to stationary motion. Under certain initial conditions one can obtain a solution of the form (5): the initial pressure distribution propagates along the pipe with constant velocity.

It is not difficult to find, by numerical integration, the functions f and h from (6)–(9). However, it is of greatest interest to obtain a solution in elementary functions. We have succeeded in finding such a solution in the form (8) for $\alpha = -(m+1)/(n+1)$:

$$\left(\frac{m+2}{n}\right)^n f^{m-n+1} + k\eta^{-n(n+1)/m+2} = \text{const}, \quad h = -\frac{n}{m+2} \eta^{-n/(m+2)} f.$$

5°. The case $k = 0$ ($m = n - 1$) is of special applied interest. The optimal system of one-parameter subgroups consists in this case of the subgroups

$$X_1, X_2 + \gamma X_1, X_4, X_3 + \alpha X_4, -X_2 + \beta X_1 + X_4, -X_1 + X_4,$$

where α, β, γ are arbitrary constants.

To the operator $X_2 + \gamma X_1$ there corresponds a solution of the form

$$u = f(x - \gamma t), \quad v = h(x - \gamma t); \quad (10)$$

to the operator $X_3 + \alpha X_4$:

$$u = t^{\alpha p} f(\eta), \quad v = t^{\alpha p - 1/(n+1)} h(\eta), \quad \eta = xt^{-p}; \quad (11)$$

to the operator $-X_2 + \beta X_1 + X_4$:

$$u = e^{-t} f(x - \beta t), \quad v = e^{-t} h(x - \beta t); \quad (12)$$

to the operator $-X_1 + X_4$:

$$u = e^{-x} f(t), \quad v = e^{-x} h(t). \quad (13)$$

6°. The functions f and h in (11) are determined as solutions of the system of ordinary differential equations

$$\alpha f - \eta f' + \frac{1}{p} h' = 0, \quad f' + \frac{h^n}{f^{n-1}} = 0. \quad (14)$$

Introducing the new unknown function $\zeta = h/f$, we obtain the system

$$\zeta' = -\alpha p - \eta p \zeta^n + \zeta^{n+1}, \quad (15)$$

$$f'/f + \zeta^n = 0. \quad (16)$$

If $\alpha = -1$, it is not difficult to find a particular solution containing an arbitrary constant c :

$$\zeta = p\eta, \quad f = c \exp \left\{ -\frac{p^n}{n+1} \eta^{n+1} \right\};$$

Fig. 1

Figure 1: Fig. 1

Fig. 2

Figure 2: Fig. 2

$$h = cp\eta \exp \left\{ -\frac{p^n}{n+1} \eta^{n+1} \right\}.$$

Fig. 1

For turbulent gas flow $n = 2$, this solution in the original variables x, t has the form

$$u = ct^{-2/3} \exp\{-4/27 x^3 t^{-2}\}, \quad v = \frac{2c}{3} x t^{-5/3} \exp\{-4/27 x^3 t^{-2}\}. \quad (17)$$

Solution (17) may be interpreted as a flow in a semi-infinite pipe caused by an explosion-like impulse of the δ -function type.

Fig. 2

The dependence $v(x)$ at times $t_1 < t_2 < t_3$ is shown in Fig. 1. Varying the value of α in (15), we obtain a series of different solutions. For $\alpha = 0$ and $\alpha = 1/n$, one can single out solutions admitting a physical interpretation: in the first case as the propagation of a pressure jump, and in the second as the propagation of a flow-rate jump through initially quiescent gas.

Fig. 2 shows the qualitative behavior of the integral curves of equation (15) for $\eta > 0$ as a function of α .

Solutions with a continuously varying flow rate, defined on the entire positive semi-axis η , admit a physical interpretation. If $a > 0$, the only case of interest for applications is the separatrix of two families of integral curves with vertical asymptotes.

The asymptotic behavior of the separatrix for large η , independently of a , is determined by the formulas

$$\xi \sim p\eta + p^{-n+1}(1+a)\frac{1}{\eta^n} + \dots,$$

$$f \sim \eta^{-(1+a)n} \exp \left\{ -\frac{p^n}{n+1} \eta^{n+1} \right\} + \dots.$$

For $a < 0$, in addition to the separatrix, an entire family of integral curves is defined for all $\eta > 0$. When η is large, the lines of this family approach the curve

$$\xi = (-a)^{1/n} \eta^{-1/n}.$$

All-Union Scientific Research
Institute of Natural Gases

Received
15 VII 1966

CITED LITERATURE

¹ L. V. Ovsyannikov, DAN, **118**, No. 3 (1958).

² L. V. Ovsyannikov, DAN, **125**, No. 3 (1959).

Note: Figure translations are in progress. See original paper for figures.

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