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Abstract

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MATHEMATICAL PHYSICS

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ON THE CLEBSCH-GORDAN COEFFICIENTS OF SYMMETRIC REPRESENTATIONS OF THE GROUPS SU_n

The importance of the theory of representations of the groups SU_n for modern theoretical physics makes its further development necessary. This applies especially to the reduction of the direct product of irreducible representations, expressed by a Clebsch-Gordan series. Questions concerning the properties of the Clebsch-Gordan coefficients (C.-G.) and methods for their rapid determination still await solution.

In this note we shall indicate a method for expressing the isoscalar factors of the C.-G. coefficients of symmetric representations of the groups SU_n through a product of $n-2$ factors, each of which depends on the parameters of representations of the groups SU_k and $U_1 \times SU_{k-1}$ ($k = 3, 4, \dots, n$). Let us recall that symmetric representations are irreducible representations with a Young diagram consisting of one row, or representations contragredient to them. The Young diagram of the latter consists of $n-1$ ($k-1$ for the subgroups SU_k) equal rows.

In solving the problem posed we shall proceed from the construction of recurrence relations for the C.-G. coefficients under consideration, since this method is the most convenient (cf., for example, $(^1, ^2)$). We begin the exposition by indicating the method of using these relations in the case of the group SU_3 .

We shall denote the basis functions of irreducible representations of the group by

$$\left| \begin{matrix} (\lambda\mu) \\ IMY \end{matrix} \right\rangle, \quad (1)$$

where $\lambda + \mu$ and μ are the lengths of the rows of the Young diagram; I and M are the isospin and its projection; Y is the hypercharge. We shall use the system of phases $(^3)$, since it facilitates the generalization of the results to the case of groups with $n > 3$. In this system of phases the matrix elements of the operators $E_{k, k-1}$ are positive. We shall denote the C.-G. coefficients by

$$\begin{bmatrix} (\lambda_1 \mu_1) & (\lambda_2 \mu_2) & (\lambda \mu)_\rho \\ I_1 M_1 Y_1 & I_2 M_2 Y_2 & I M Y \end{bmatrix} = \begin{bmatrix} (\lambda_1 \mu_1) & (\lambda_2 \mu_2) & (\lambda \mu)_\rho \\ I_1 Y_1 & I_2 Y_2 & I Y \end{bmatrix} \begin{bmatrix} I_1 & I_2 & I \\ M_1 & M_2 & M \end{bmatrix}. \quad (2)$$

The first factor on the right-hand side represents the isoscalar factor, and the second the C.-G. coefficients of the group SU_2 . Here $M = M_1 + M_2$, $Y = Y_1 + Y_2$. The parameter ρ , distinguishing repeated representations, will not be needed by us. We remove the remaining phase indeterminacy of the C.-G. coefficients by considering their value positive at $I_2 = 0$.

For symmetric representations λ or μ is equal to zero. The two representations being coupled may be of identical or different contragredientness. The expressions for the isoscalar factors depend on this. Using the relation obtained when the infinitesimal operator E_{32} acts on a basis function of the coupled representations, we obtain the following recurrence relation:

$$\begin{aligned} & \begin{bmatrix} (\lambda_1 0) & (\lambda_2 0) & (\lambda_1 + \lambda_2 - 2s, s) \\ I_1 & I_2 & I + 1 \end{bmatrix} [(I_1 + I_2 + I - s + 1)(\lambda_1 + \lambda_2 - s - I_1 - I_2 - I)]^{1/2} = \\ & = \begin{bmatrix} (\lambda_1 0) & (\lambda_2 0) & (\lambda_1 + \lambda_2 - 2s, s') \\ I_1 - \frac{1}{2} & I_2 & I + \frac{1}{2} \end{bmatrix} [(I_1 - I_2 + I)(\lambda_1 - 2I_1 + 1)]^{1/2} + \\ & + \begin{bmatrix} (\lambda_1 0) & (\lambda_2 0) & (\lambda_1 + \lambda_2 - 2s, s') \\ I_1 & I_2 - \frac{1}{2} & I + \frac{1}{2} \end{bmatrix} [(I - I_1 + I_2)(\lambda_2 - 2I_2 + 1)]^{1/2}. \quad (3) \end{aligned}$$

Comparing this formula with formula (12.3) for the Clebsch-Gordan coefficient of SU_2 from (3), we find that they are isomorphic. By changing the constraints on I and M we obtain another recurrence relation, isomorphic to which one may be obtained by combining (16.18) and (12.3) from (4). The corresponding formulas for the isoscalar factors of SU_3 and the Clebsch-Gordan coefficients of SU_2 coincide if one makes the substitution

$$\begin{aligned} & \begin{bmatrix} (\lambda_1 0) & (\lambda_2 0) & (\lambda_1 + \lambda_2 - 2s, s) \\ I_1 & I_2 & I \end{bmatrix} = \\ & = \begin{bmatrix} \frac{\lambda_1 - I_1 - I_2 + I}{2} & \frac{\lambda_2 - I_1 - I_2 + I}{2} & \frac{\lambda_1 + \lambda_2}{2} - s \\ \frac{I + 3I_1 - I_2 - \lambda_1}{2} & \frac{I - I_1 + 3I_2 - \lambda_2}{2} & I_1 + I_2 + I - \frac{\lambda_1 + \lambda_2}{2} \end{bmatrix} = (4a) \\ & = \begin{bmatrix} I & \frac{\lambda_1 + \lambda_2}{2} - I_1 - I_2 & \frac{\lambda_1 + \lambda_2}{2} - s \\ I_1 - I_2 & \frac{\lambda_1 - \lambda_2}{2} - I_1 + I_2 & \frac{\lambda_1 - \lambda_2}{2} \end{bmatrix}. \quad (4) \end{aligned}$$

Here Y_i ($i = 1, 2$) are omitted, since they are expressed as

$$Y_i = 2I_i - \frac{2}{3}\lambda_i. \quad (4)$$

(4a) and (4) are Clebsch-Gordan coefficients of the group SU_2 , with the second obtained from the first by means of Regge symmetry properties for Wigner coefficients.

It is easy to verify that (4a) is normalized with respect to summation over I_1 and I_2 for fixed I and Y . On the other hand, (4) is normalized with respect to summation over the parameters of the third column, if it is first multiplied also by a Clebsch-Gordan coefficient of SU_2 . It should be noted that from (4) one can obtain other types of substitutions by means of the symmetry properties of the Clebsch-Gordan coefficients of the group SU_3 ⁽⁵⁾.

In the case of different contragredientnesses, instead of (3) we obtain

$$\begin{aligned} & \begin{bmatrix} (0\mu) & (\lambda 0) & (\lambda - a, \mu - a) \\ I_1 & I_2 & I \end{bmatrix} = \\ & = (-1)^{I_1+I_2-I} \begin{bmatrix} (\lambda 0) & (0\mu) & (\lambda - a, \mu - a) \\ I_2 & I_1 & I \end{bmatrix} = \\ & = \begin{bmatrix} \frac{\mu - I_1 + I_2 - I}{2} & \frac{I_1 - I_2 + I + \lambda + 1}{2} & \frac{\lambda + \mu + 1}{2} - a \\ \frac{\mu - 3I_1 - I_2 + I}{2} & \frac{I_1 + 3I_2 + I - \lambda + 1}{2} & \frac{\mu - \lambda + 1}{2} + I - I_1 + I_2 \end{bmatrix}, \end{aligned} \quad (5a)$$

where

$$Y_1 = \frac{2\mu}{3} - 2I_1, \quad Y_2 = 2I_2 - \frac{2}{3}\lambda; \quad (5)$$

(5a), (5) are also Clebsch-Gordan coefficients of the group SU_2 , whose recurrence relations are known. However, the relation corresponding to formula (3) has a somewhat different form.

The formulas given make it easy to obtain the formulas of Section 5 of ⁽⁶⁾, and our expression (4) is considerably simpler than (7.9) of ⁽⁶⁾.

In the case of the groups SU_n , we shall denote the basis functions by a set of Gel'fand-Biedenharn parameters ^(7,3). In the case of symmetric representations this set will be the following:

$$\begin{pmatrix} h_n & 0 & 0 \dots & 0 \\ & h_{n-1} & 0 \dots & \\ & & h_{n-2} & 0 \dots \end{pmatrix} \quad (6a)$$

The basis functions connected from two such representations may be denoted by

$$\left(\begin{array}{ccccccc} h_n + h'_n - s_n & & & & s_n & 0 & 0 \dots 0 \\ & h_{n-1} + h'_{n-1} - s_{n-1} & & & & s_{n-1} & 0 \dots 0 \\ & & h_{n-2} + h'_{n-2} - s_{n-2} & & & & s_{n-2} \dots 0 \\ & & \dots & & \dots & \dots & \dots \end{array} \right) \quad (6)$$

We denote the separate factor in the C.-G. coefficients for the reduction of SU_k to $U_1 \times SU_{k-1}$ by

$$\left[\left(\begin{array}{ccc} h_k & 0 & 0 \dots \\ h_{k-1} & 0 \dots & \end{array} \right) \left(\begin{array}{cc} h'_k & 0 \dots \\ h'_{k-1} & \dots \end{array} \right) \left(\begin{array}{ccc} h_k + h'_k - s_k & s_k & 0 \dots \\ & h_{k-1} + h'_{k-1} - s_{k-1} & s_{k-1} \dots \end{array} \right) \right] =$$

$$= H_k = \left(\begin{array}{cc|c} h_k & h'_k & s_k \\ h_{k-1} & h'_{k-1} & s_{k-1} \end{array} \right). \quad (7)$$

Acting with the operator $E_{k,k-1}$ on functions of the type (6a) and (6), we ascertain that the expressions for the matrix elements of this operator, according to formula (60) from (3), do not depend on the parameters $(k-i)$ ($i = 3, 4, \dots$), nor on k in explicit form.

In the case of identical contragredients we obtain recurrence relations as for the representations of the groups SU_3 . Without loss of generality, on the basis of the independence, indicated in (3), of the separate factor of the C.-G. coefficients from the parameters $k-i$ ($i = 2, 3, \dots$) of the rows, we may assume $s_{k-i} = 0$. Then the expression for the matrix elements of the operator $E_{k,k-1}$ for any k coincides with the expression for the matrix element E_{32} of the group SU_3 , where the parameters of the k -th row stand in place of the 3rd row, $k-1$ in place of the 2nd, etc. If we take the expression for the separate factor of the C.-G. coefficient

$$H_k \left(\begin{array}{cc|c} h_k & h'_k & s_k \\ h_{k-1} & h'_{k-1} & s_{k-1} \end{array} \right) =$$

$$= \left[\begin{array}{ccc} \frac{h_k - s_{k-1}}{2} & \frac{h'_k - s_{k-1}}{2} & \frac{h_k + h'_k}{2} - s_k \\ h_{k-1} - \frac{h_k + s_{k-1}}{2} & h'_{k-1} - \frac{h'_k + s_{k-1}}{2} & h_{k-1} + h'_{k-1} - s_{k-1} - \frac{h_k + h'_k}{2} \end{array} \right], \quad (8)$$

which holds for $k = 2$ and 3 , and assume that it is valid for the subgroup SU_{k-i} ($i = 1, 2, \dots$), then in the recurrence relation H_{k-1} will play the role of the C.-G.

coefficient of the group SU_2 in the relation for SU_3 . In this case the factors H_{k-i} ($i = 2, 3, \dots$) are simplified, and the relation for H_k coincides with the same relation for the group SU_3 , if only the basis functions of irreducible representations of the latter are denoted by a set of Gelfand–Biedenharn parameters.

In generalizing (5) to the case of the groups SU_n , we denote the basis function contragredient to the function (6a) by

$$\begin{pmatrix} 0 \dots 0 & 0 & & g_{-n} \\ & \dots & 0 & -g_{n-1} \\ & & \dots & -g_{n-2} \\ & & & \dots \end{pmatrix}, \quad g_k \geq 0. \quad (9)$$

Relating representations (9) and (6a) (by reducing the direct product), we obtain basis functions of the type

$$\begin{pmatrix} h_n - a_n & 0 & 0 & \dots & 0 & a_n - g_n \\ & h_{n-1} - a_{n-1} & 0 & \dots & 0 & a_{n-1} - g_{n-1} \\ & & h_{n-2} - a_{n-2} & \dots & a_{n-2} - g_{n-2} & \end{pmatrix}. \quad (10)$$

Obviously, by adding g_n to all parameters in (9), or $g_n - a_n$ in (10), we would get rid of negative parameters; however, for our purposes the forms (9) and (10) are more convenient. We denote the corresponding separate factor of the Clebsch–Gordan coefficient by

$$G_k \left(\begin{array}{cc} g_k & h_k \\ g_{k-1} & h_{k-1} \end{array} \parallel \begin{array}{c} a_k \\ a_{k-1} \end{array} \right). \quad (11)$$

If the operator $E_{k,k-1}$ is applied to functions of the type (10) and one takes $a_{k-2} = g_{k-2}$, it is found that the matrix elements of this operator can be obtained from the matrix elements of the operator E_{32} of the group SU_3 . For this purpose one should make the substitutions

$$\lambda - a \rightarrow h_k - a_k + k - 3, \quad \mu - a \rightarrow g_k - a_k,$$

$$2I \rightarrow h_{k-1} - g_{k-1} - 2a_{k-1} + k - 3. \quad (12a)$$

Applying the operator $E_{k,k-1}$ to basis functions of the type (9) and (6a), we are led to the equivalence

$$\lambda - 2I_2 \rightarrow h_k - h_{k-1}, \quad \mu - 2I_1 \rightarrow g_k - g_{k-1}. \quad (12b)$$

If

$$G_k \left(\begin{array}{cc|c} g_k & h_k & a_k \\ g_{k-1} & h_{k-1} & a_{k-1} \end{array} \right) =$$

$$= \left[\begin{array}{ccc} \frac{g_k - g_{k-1} + a_{k-1}}{2} & \frac{h_k + g_{k-1} + a_{k-1} + k - 2}{2} & \frac{g_k + h_k + k - 2}{2} - a_k \\ \frac{g_k - g_{k-1} - a_{k-1}}{2} & h_{k-1} + \frac{g_{k-1} - h_k - a_{k-1} + k - 2}{2} & h_{k-1} - a_{k-1} + \frac{g_k - h_k + k - 2}{2} \end{array} \right] \quad (13)$$

is valid for G_{k-i} ($i = 1, 2, \dots$), then by substituting G_{k-i} into the corresponding recurrence relation we obtain a mutual correspondence with the recurrence relation for the group SU_3 , provided only that in (12) one regards $a \rightarrow a_n$. Therefore, if (13) is valid for SU_3 , it is also valid for SU_k for any $k = 3, 4, \dots$. It should be noted that the symmetry of (5) and (13) with respect to the parameters g and h (λ and μ) (see (5)) is visible when the Clebsch-Gordan coefficients are expressed in terms of Wigner coefficients with Racah parameters.

From what has been set forth it follows that the indicated method makes it possible to express the Clebsch-Gordan coefficients of symmetric representations of the group SU_n as

$$\prod_{k=2}^n C_k,$$

where C_2 is the Clebsch-Gordan coefficient of the group SU_2 , and C_k , $k \geq 3$, are respectively the factors H_k or G_k , equal to the Clebsch-Gordan coefficients of the group SU_2 .

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