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Abstract

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MATHEMATICAL PHYSICS

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THE USE OF EVOLUTION GRAPHS OF FUNCTIONAL NETWORKS IN SOLVING BOUNDARY-VALUE PROBLEMS OF ELECTRODYNAMICS BY THE METHOD OF MATCHING

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1. Let a volume V , in which Maxwell's equations for the complex amplitudes E and H hold, be divided into partial regions $i = 1, 2, 3, \dots, P$. The solution of the first boundary-value problem of electrodynamics ^(1,3) for a region i adjacent to other regions i' along surfaces $S_j^{(i)}$, $j = 1, 2, 3, \dots, M^{(i)}$, establishes a unique functional relation between the tangential components of the fields $E_k^{(i)}$ and $H_j^{(i)}$ on $S_j^{(i)}$:

$$\{H_j^{(i)}\}_{j=1}^M = (Y_{jk}^{(i)}) \{E_k^{(i)}\}_{k=1}^M, \quad (1)$$

where $(Y_{jk}^{(i)})$ is an $M \times M$ matrix composed of the linear conductivity operators ⁽¹⁾. The continuity equations for E_k and H_j on $S_j^{(i)}$ will be

$$E_{k_m}^{(i'')} = \mathcal{E} H_{j_m}^{(i')}, \quad m = 1, 2, \dots, R, \quad (2)$$

where $\mathcal{E}$ is the identity-transformation operator. The set of partial regions is a functional network of $M \times M$ cells (M input and M output channels), connected by ideal links described by the system of operator equations (1), (2). The solution of the first boundary-value problem for the volume V is now reduced to the solution of the network equations. Having represented (1), (2) by a two-color evolution graph according to the rules given in ⁽²⁾, we carry out the investigation and solution of (1), (2) graphically.

2. Consider the case of a partition (Fig. 1a) in which the network is a chain of adjacently interacting 2×2 cells, described by 2×2 matrices composed of conductivity operators $Y_{11}^{(i)}$, $Y_{12}^{(i)}$, $Y_{21}^{(i)}$, and $Y_{22}^{(i)}$, $i = 1, 2, 3, \dots, R-1$. At the ends of the network two-channel cells $i = 0$ and $i = R$ are connected,

with input-conductivity operators $Y^{(0)} = [Z^0]^{-1}$ and $Y^{(R)} = [Z^{(R)}]^{-1}$. A time-harmonic ($e^{-i\omega t}$) external action is localized in the section S_{i+1} and is specified by the amplitude functions E_0 and H_0 . The evolution graph of such a network is shown in Fig. 1b. For its construction an inversion of the arcs $Y_{11}^{(i)}$, $PR - (P = 2, R = 2)$ of the graphs of all cells, $i = 1, 2, \dots, R-1$, has been carried out according to the rules ⁽²⁾. The right half of the graph in Fig. 1b has one path connecting the vertices $4^{(i+1)}$ with $1^{(i+1)}$ ($42142 \dots 1421 - 431431431 \dots 43141$), transmitting states from left to right and then back. The left half of the graph in Fig. 1b also has one path connecting the vertices $3^{(i)}$ with $2^{(i)}$ ($314314 \dots 431 - 421421 \dots 4 \dots 42$), transmitting states from right to left and back. Both paths are loaded with feedback loops, and for computing the operators of the loaded paths ⁽²⁾ it is convenient to apply the recurrence formula (3), obtained directly from the graph of Fig. 1b, where $\tilde{Y}_i$ is the path operator between the vertices $1^{(i)}$ and $4^{(i)}$, and $\tilde{Y}_{i+1}$ is that between $1^{(i+1)}$ and $4^{(i+1)}$,

$$\tilde{Y}_i = Y_{11}^{(i)} + Y_{21}^{(i)} [\tilde{Y}_{i+1} - Y_{22}^{(i)}]^{-1} Y_{21}^{(i)}. \quad (3)$$

Let us denote the obtained path operators between $1^{(i+1)}$ and $4^{(i+1)}$, and between $3^{(i)}$ and $2^{(i)}$, respectively, by $\tilde{Y}_i^{(R)}$ and $\tilde{Y}_i^{(0)}$ (these are $Y^{(R)}$ and $Y^{(0)}$, recalculated from the ends of the chain in Fig. 1a to the section $i, i+1$). Now the graph in Fig. 1b can be replaced by the graph in Fig. 1c, and with its aid one can determine the 2×2 matrices of the operators Z_1 and Z_2 , relating

$$\begin{pmatrix} E_1^{(i+1)} \\ H_1^{(i+1)} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} E_2^{(i)} \\ H_2^{(i)} \end{pmatrix} \quad \text{to} \quad \begin{pmatrix} E_0 \\ H_0 \end{pmatrix};$$

$$Z_1 = \begin{pmatrix} -\tilde{Z}_i^{(R)} \Pi'' \tilde{Y}_i^{(0)} & \tilde{Z}_i^{(R)} \Pi'' \\ \Pi'' \tilde{Y}_i^{(0)} & -\Pi'' \end{pmatrix}, \quad Z_2 = \begin{pmatrix} \Pi' & \Pi' \tilde{Z}_i^{(R)} \\ \tilde{Y}_i^{(0)} \Pi' & \tilde{Y}_i^{(0)} \Pi' \tilde{Z}_i^{(R)} \end{pmatrix}; \quad (4)$$

here $\Pi'' = (\mathcal{E} + \tilde{Y}_i^{(0)} \tilde{Z}_i^{(R)})^{-1}$ and $\Pi' = (\mathcal{E} + \tilde{Z}_i^{(R)} \tilde{Y}_i^{(0)})^{-1}$ are the feedback-loop operators of the circuit 6145326 with contour operators $(-\tilde{Y}_i^{(0)} \tilde{Z}_i^{(R)})$ and $(-\tilde{Z}_i^{(R)} \tilde{Y}_i^{(0)})$, counted from vertices 5 for Π' and 6 for Π'' . The graph in Fig. 1c gives the solution for forced oscillations in the volume V . They exist if the operator of the circuit 6145326 differs from $\mathcal{E}$, which follows from (2). Otherwise, natural oscillations exist. To find the fields $E^{(i)}$ and $H^{(i)}$ in any section i , let us replace, by inverting the arcs, the matrices of the conductance operators $(Y_{jk}^{(i)})$ (the graph in Fig. 1b) by the Breisig matrices $(A_{jk}^{(i)})$ (1), explicitly relating $E_2^{(i)}$ and $H_2^{(i)}$ to $E_1^{(i)}$ and $H_1^{(i)}$.

Fig. 1

Fig. 1

Figure 1: Fig. 1

Combining the states $E_1^{(i)}$ and $H_1^{(i)}$ into one, determined by the vector-function

$$\left\| \begin{array}{c} E_1 \\ H_1 \end{array} \right\|,$$

we replace the graph in Fig. 1b by the simpler graph in Fig. 1d. With its help it is easy to construct expressions relating

$$\left\| \begin{array}{c} E_1^{(i')} \\ H_1^{(i')} \end{array} \right\|$$

to $E_0^{(i+1)}$ and $H_0^{(i+1)}$ in any section $i', i' + 1$ (see (1)).

3. If the cells into which the volume V is divided are identical and are loaded at the ends by the cells 0 and R with Y^0 and Y^R , determined from equation (3), where

$\widetilde{Y}_i = \widetilde{Y}_{i+1}$ (for the case $Y_{11} = Y_{22}$ and $Y_{12} = Y_{21}$, $Y^{0,R} = Y_{11} \pm Y_{12}$), then it makes sense to carry out further investigation of the chain by the method of normal waves [4]. The problem of finding the normal waves can be reduced to the problem of the natural oscillations of a cell closed by a conditional waveguide according to the periodicity condition

$$E_2 = -e^{-i\psi} E_1, \quad H_2 = -e^{-i\psi} H_1.$$

As a result we obtain the closed graph of Fig. 2a. Reading from it the system of equations for E_1 , we obtain

$$\begin{aligned} (Y_{11} + Y_{22})E_1 &= \\ &= (e^{i\psi}Y_{21} + e^{-i\psi}Y_{12})E_1, \end{aligned}$$

where Y_{ik} are the conductance operators of the cell.

4. Let us apply the stitching method to such a cell and divide it into partial regions (for example, according to Fig. 2b) with the prescribed matrices of the conductance operators (1). Joining the cells $i = I$ and $i = II$ according to the periodicity and continuity conditions, we obtain the graph of Fig. 2c. It consists of a single isolated contour. Computing the operator of this contour and equating it to $\mathcal{E}$, we obtain a system of homogeneous equations

$$\begin{pmatrix} Y_{11}^{(I)} - Y_{22}^{(II)} & Y_{12}^{(I)} + e^{i\psi} Y_{21}^{(II)} \\ Y_{21}^{(I)} + Y_{12}^{(II)} e^{-i\psi} & Y_{22}^{(I)} - Y_{11}^{(II)} \end{pmatrix} \times \begin{pmatrix} E_1 \\ E_2 \end{pmatrix} = 0 \quad (5)$$

E_1 and E_2 are functions of the fields E_τ in sections $\langle 1 \rangle$ and $\langle 2 \rangle$. In Fig. 2c another division of the cell into partial regions I and II is given. Region I corresponds to a 6-channel, and II to a 2-channel, cell of the network described by the matrices (1). The graph of such a division is presented in Fig. 2d. It also consists of a single isolated contour. Equating the operator of this contour to $\mathcal{E}$, we obtain an equation for E_3 in section $\langle 1 \rangle$:

$$\begin{aligned} & [(Y_{11} - Y_{22} - e^{-i\psi} Y_{12} + e^{i\psi} Y_{21})(Y_{31} - e^{-i\psi} Y_{32})^{-1}(Y_{00} - Y_{33}) + \\ & + (Y_{13} + e^{i\psi} Y_{33})] E_3 = 0. \end{aligned} \quad (6)$$

(5) and (6) are valid for regions I and II of arbitrary shape. The conditions of their compatibility are the dispersion equations relating the wave numbers of the normal waves ψ with the frequency ω : $\psi = f[\omega]$. For numerical calculations they must be algebraized. If I and II are cylindrical waveguide sections with arbitrary contours Γ_I and Γ_{II} of the cross sections (Fig. 2e), then, introducing the coordinates r, φ, z , we represent $(Y_{j,k}^{1,2})$ by diagonal matrices in the basis of normal waves traveling along the z -axis. In this case E and H are expanded in terms of the forms of the normal waves $e_{m,n}^{(1)}(r, \varphi)$ and $e_{m,n}^{(2)}(r, \varphi)$, traveling along z in I and II with wave numbers $\alpha_{m,n}$ and $\gamma_{m,n}$, respectively. If the fields E_1 and E_2 are approximated by Fourier series in the basis $\{\mathcal{E}_s\}_s$, different from $\{e_{m,n}\}_{m,n}$, i.e.

$$E_1 = \sum_{s=1}^N K_s^{(1)} \mathcal{E}_s \quad \text{and} \quad E_2 = \sum_{s=1}^N K_s^{(2)} \mathcal{E}_s(r, \varphi),$$

then the scalar pro-

admittances $(Y_{j,k} \mathcal{E}_s, \mathcal{E}_{s'})$, which are elements of the matrices $\|Y_{jk}\|$, will be convolutions over the indices m and n (see formula (8)). The dispersion equation of the waveguide composed of the cells of Fig. 2e, in matrix form, will be

$$\det \begin{bmatrix} a - \alpha - \beta \cos \psi & -\beta \sin \psi \\ -\beta \sin \psi & a + \alpha + \beta \cos \psi \end{bmatrix} = 0, \quad (7)$$

where a , α , and β are square $N \times N$ matrices with elements

$$a_{s,s'} = \sum_{m,n} \frac{\text{cth } \gamma_{m,n} d}{\gamma_{m,n}} \beta_{m,n,s} \beta_{m,n,s'} + \sum_{m,n} \frac{\text{cth } \alpha_{m,n} \tau}{\alpha_{m,n}} \alpha_{m,n,s} \alpha_{m,n,s'}, \quad (8)$$

$$\alpha_{s,s'} = \sum_{m,n}^{\infty} \frac{\alpha_{m,n,s} \alpha_{m,n,s'}}{\alpha_{m,n} \operatorname{sh} \alpha_{m,n} \tau}, \quad \beta_{s,s'} = \sum_{m,n}^{\infty} \frac{\beta_{m,n,s} \beta_{m,n,s'}}{\gamma_{m,n} \operatorname{sh} \gamma_{m,n} d},$$

where $\alpha_{m,n,s}$ and $\beta_{m,n,s}$ are the Fourier coefficients of the expansion of $\mathcal{E}_s$ in $e_{m,n}^{(1)}$ and $e_{m,n}^{(2)}$, respectively. Below we consider cases in which branches I and II have circular cross sections (Fig. 2). Restricting ourselves to the case of axially symmetric waves of type TH, we obtain, for $\alpha_{m,n,s} = \alpha_{m,s}$ and $\beta_{m,n,s} = \beta_{m,s}$, the following results.

If $\{\mathcal{E}_s\}$ coincides with one of the bases of normal waves, for example with $\{e_m(r/a)\}$, then

$$\alpha_{m,s} = \delta_{m,s} = \begin{cases} 1, & m = s, \\ 0, & m \neq s; \end{cases} \quad \beta_{m,s} = \frac{2ab\rho_{0,m}J_0(\rho_{0,m}a/b)J_1(\rho_{0,s})}{J_1^2(\rho_{0,m})(\rho_{0,s}^2b^2 - \rho_{0,m}^2a^2)}. \quad (9)$$

To take into account the electrostatic singularity of E at the sharp edges of the cell, we choose the basis

$$\mathcal{E}_s = (r/a)/[1 - (r/a)^2]^{1/p};$$

then

$$\beta_{m,s} = \frac{2\sqrt{\pi a^2}}{b|J_1(\rho_{0,m})|} \sum_{r=0}^{s-1} (-1)^r C_r^r 2^{r-1/p} \Gamma\left(r - \frac{1}{p} + 1\right) I_{r-1/p+2}\left(\rho_{0,m} \frac{a}{b}\right). \quad (10)$$

$$\alpha_{m,s} = \beta_{m,s} \quad (b = a).$$

For the case of subdividing the cell according to Fig. 2, when $Y_{12} = Y_{21}$, $Y_{23} = Y_{32}$, $Y_{13} = Y_{31}$, $Y_{11} = Y_{22}$, $Y_{13} = Y_{23}$, and equation (6) takes the form $[Y_{13}Y_{12}^{-1}Y_{13} - Y_{33} + Y_0]E_3 = 0$, it is convenient in region I to use as a basis the forms of normal waves traveling along r , with wave forms $\exp\{-i\beta_m z\}$, $\beta_m = (\psi + 2\pi m)/D$ (wave numbers $\mu_m = \sqrt{k_0^2 - \beta_m^2}$), and for II the basis of the forms of normal waves $\cos \pi p z/d$, likewise traveling along r , with wave numbers $\chi_p = \sqrt{k_0^2 - (p\pi/d)^2}$. As a result of algebraization, we obtain the dispersion equation in the form

$$\det \left[\sum_{m=-\infty}^{\infty} \frac{J_1(\mu_m a)}{\mu_m J_0(\mu_m a)} \beta_{m,s} \beta_{m,s'} - \sum_{p=0}^{\infty} \frac{D(a', b)}{\chi_p D(a, b)} \alpha_{p,s} \alpha_{p,s'} \right] = 0, \quad (11)$$

where $D(a, b)$ and $D(a', b)$ are given in (5).

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Note: Figure translations are in progress. See original paper for figures.

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