

APPROXIMATELY MINIMAX DETECTION OF A VECTOR SIGNAL ON A GAUSSIAN BACKGROUND

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Abstract

Full Text

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MATHEMATICS

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APPROXIMATELY MINIMAX DETECTION OF A VECTOR SIGNAL ON A GAUSSIAN BACKGROUND

This note adjoins the note ⁽²⁾. A signal is considered in the form of a p -dimensional column vector ξ , and the background in the form of a normal vector $X - \xi \in N(0, \Sigma)$, where Σ is an unknown nondegenerate correlation matrix. A repeated sample $X_1, \dots, X_N$ is taken. The hypothesis of absence of signal $H_0 : \xi = 0$ is tested against the composite hypothesis $H_1 : N\xi^T \Sigma^{-1} \xi = \delta$, and also against the composite hypothesis $H'_1 : N\xi^T \Sigma^{-1} \xi \geq \delta$, where $\delta > 0$ is a given number.

The expression $N\xi^T \Sigma^{-1} \xi$ may be interpreted as the relative energy of the signal, and the two problems posed—as problems of detecting a signal of prescribed relative energy δ , or of relative energy not less than δ . We seek a minimax procedure for signal detection, i.e., such a (randomized) test Φ for which, at a given level α , the minimum power on the composite alternative H_1 (or H'_1) is maximal among all tests Φ of level $\leq \alpha$.

The works of many authors (see ⁽¹⁾) lead one to suppose that such a test is the well-known Hotelling test, based on the statistic

$$T^2 = N(N-1)\bar{X}^T S^{-1} \bar{X}. \quad \text{Here } \bar{X} = \frac{1}{N} \sum_{i=1}^N X_i, \quad S = \sum_{i=1}^N (X_i - \bar{X})(X_i - \bar{X})^T.$$

The test has the form $T^2 \geq T_0^2$. In ⁽¹⁾ this supposition was proved for testing H_0 against H_1 for $p = 2$ and $N = 3$; in ⁽²⁾, for testing H_0 against H_1 and H'_1 for $p = 2$ and $N = 4$. Further progress encounters great analytical difficulties.

In the present note the requirement of minimaxity is replaced by the weaker requirement of “ ε -minimaxity” of a test: for any $\varepsilon > 0$, its minimum power on the alternatives H_1 or H'_1 , at a given level α and sample size $N \geq N_0(\varepsilon)$, must differ from the maximum of the minimum powers over all tests of level $\leq \alpha$ by no more than ε .

Of course, for a given level $\alpha \in (0, 1)$ and increasing N , only weak signals are of interest, for which $\delta = N\xi^T \Sigma^{-1}$ can grow only slowly; otherwise the power will tend to 1, and the assertion of ε -minimaxity will become trivial.

Theorem 1. *The Hotelling test $\Phi_N : T^2 \geq T_0^2$ for testing the hypothesis H_0 against the hypothesis H_1 is ε -minimax for every level $\alpha \in (0, 1)$: for any $\varepsilon > 0$ the relation holds*

$$\sup_{\Phi} \inf_{\theta \in H_1} E_{\theta} \Phi - \inf_{\theta \in H_1} E_{\theta} \Phi_N \leq \varepsilon \quad (1)$$

for $N > N_0(\varepsilon)$.

Here $\theta = (\xi, \Sigma)$ is the parameter (ξ is the parameter under study, Σ the nuisance parameter); Φ ranges over all tests of level $\leq \alpha$; E_{θ} denotes the sign of mathematical expectation.

* In this form Theorem 1 can apparently also be derived from A. Wald's theorems (see ⁽³⁾).

More precise information is given by Theorem 1'.

Theorem 1'. If the level $\alpha = \alpha_N$ is subject to the conditions

$$O(\exp -(\ln N)^{1/6}) \leq \alpha \leq 1 - O(\exp -(\ln N)^{1/6}) \quad (2)$$

and we have

$$\exp [-(\ln N)^{1/6}] \leq \delta \leq (\ln N)^{1/6}, \quad (3)$$

then

$$\sup_{\Phi} \inf_{\theta \in H_1} E_{\theta} \Phi - \inf_{\theta \in H_1} E_{\theta} \Phi_N = O_{\varepsilon} \left(\frac{1}{N^{1-\varepsilon}} \right) \quad (4)$$

for any $\varepsilon > 0$.

Here Φ ranges over all tests of level $\leq \alpha$.

Theorem 2. The Hotelling test $\Phi_N : T^2 \geq T_0^2$ for testing H_0 against the composite hypothesis H'_1 is ε -minimax for any fixed level $\alpha \in (0, 1)$ and $\delta > 0$: under the conditions of Theorem 1, relation (1) holds with H_1 replaced by H'_1 .

The proof of these theorems is based on the method of N. Giri, J. Kiefer, and C. Stein, developed in their paper ¹. The central place in it is occupied by a lemma on the Giri–Kiefer–Stein integral equation (see ¹, p. 1530). This equation has the form

$$\int_{\Gamma_1} \exp \left\{ \gamma_1 \sum_{j=1}^p \tau_j \sum_{i>j} \beta_i \right\} \prod_{i=1}^p \Phi \left(\frac{N-i+1}{2}, \frac{1}{2} \gamma_1 \beta_i \tau_i \right) \times \lambda(\beta_1, \dots, \beta_p) d\beta = \Phi \left(\frac{N}{2}, \frac{p}{2}, \gamma_1 \right), \quad (5)$$

i.e. it is a Fredholm equation of the first kind.

Here Γ_1 is a simplex; $\beta_i \geq 0$, $\sum_{i=1}^p \beta_i = 1$; $\gamma_1 > 0$ is a parameter; $\tau_i \geq 0$, $\sum_{i=1}^p \tau_i = 1$, τ_i are variables; $\lambda(\beta_1, \dots, \beta_p)$ is the desired function, $d\beta$ is an element of Lebesgue measure on the simplex Γ_1 ;

$$\Phi(a, c, x) = \sum_{j=0}^{\infty} \frac{\Gamma(a+j)\Gamma(c)}{\Gamma(a)\Gamma(c+j)} \frac{x^j}{j!}$$

is the confluent hypergeometric function (E. Kummer's series). The main lemma states:

Lemma. As $N \rightarrow \infty$, $\gamma_1 = 2\psi/N$ and $0 \leq \psi \leq 2(\ln N)^{1/3}$, the Giri–Kiefer–Stein equation (5) has the approximate solution

$$\lambda_{\infty}(\beta_1, \dots, \beta_p) = \frac{\Gamma(p/2)}{(\Gamma(1/2))^p} (\beta_1 \beta_2 \dots \beta_p)^{-1/2}, \quad (6)$$

which is a probability density on the simplex Γ_1 . Namely, when (6) is substituted into the left-hand side of (5), one obtains a residual with the right-hand side of order $O_{\varepsilon}(N^{-1+\varepsilon})$ for any $\varepsilon > 0$.

Further refinement of this method significantly improves this order and, at the same time, Theorem 1' (formula (4)).

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Note: Figure translations are in progress. See original paper for figures.

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